



Introduction to the Euler Characteristic

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Abstract

The Euler characteristic $\chi(X)$ for a finite CW complex X is defined as the alternating sum of number of cells in each dimension: $\sum_i (-1)^i (\#i\text{-cells})$. In this senior project, we prove the homotopy invariance of $\chi(X)$. We present the problem with proceeding through the combinatorial definition and motivate the necessity for the algebraic framework of homology. We prove various properties of singular homology; including homotopy invariance, while utilizing tools of homological algebra. We deduce results on degrees of maps $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$. Utilizing this, we then construct cellular homology for CW complexes and establish its isomorphism with singular homology. Culminating in the proof of the Euler-Poincaré Theorem, the characteristic is redefined as the alternating sum of Betti numbers, $\chi(X) = \sum_i (-1)^i \text{rank } H_i(X)$, confirming the homotopy invariance of $\chi(X)$.

1 Introduction

1.1 Historical Development

In between Thales (c. 624–547 BCE) and Apollonius (c. 262–190 BCE), the Greek school of mathematics, notably Pythagoras, Plato, Euclid, Archimedes, Zeno had produced an immense geometric theory. One notable instance of their study is the theorem stating that there are exactly five regular polyhedra, later known as the Platonic solids ([Ri19], pp. 1-35) and its proof is given in the final book of Euclid's *Elements*.

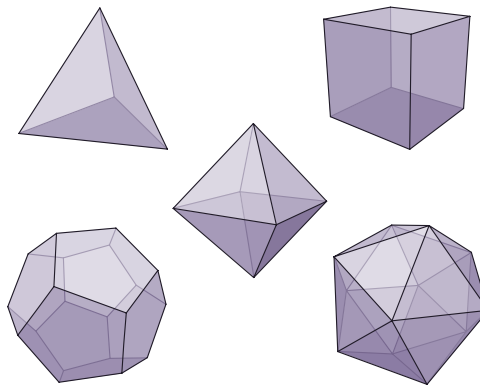


Figure 1: The 5 Platonic solids [Wi19]

The Greek school's fascination with geometry was so grand that the natural philosophy theory of atomism, of Plato, attempts to explain the physical universe by relating the

atoms to these five solids ([Ri19], pp. 31-43). There were other mathematicians who shared this fascination. For instance, these solids were later used in an attempt to model the solar system, by Kepler, in 1595, whose interest in polyhedral geometry continued until the end of his life ([Ri19], pp. 51-62).

Despite all of this brilliant mathematics of ~ 2000 years, regarding polyhedra, no one¹ was able to see a very simple formula.

In his 1750 paper, titled *Elementa Doctrinae Solidorum*, Leonhard Euler writes the following [Eu53]:

PROPOSITIO IV.	PROPOSITION IV.
§.33. In omni solido hedris planis incluso aggregatum ex numero angulorum solidorum et ex numero bedrarum binario excedit numerum acierum.	§.33. In every solid enclosed by plane faces the sum of the number of faces and the number of solid angles exceeds by two the number of edges.
DEMONSTRATIO.	DEMONSTRATION.
Scilicet si ponatur vt hactenus: numerus angulorum solidorum = S numerus acierum = A numerus hedrarum = H demonstrandum est, esse $S+H=A+2$.	Namely, if it is put as before: Number of vertices = S Number of edges = A Number of faces = H Must be demonstrated that $S+H=A+2$.
Fateri equidem cogor, me huius theorematis demonstrationem firmam adhuc eruere non potuisse.	I am forced to admit that I have not yet been able to extract a firm demonstration of this theorem.

On the left is the original Latin script and on the right the English translation. Thus, we have a famous formula of Euler for convex polyhedra.²

Theorem 1.1 (Euler's Polyhedron Formula). A polyhedron with V vertices, E edges, and F faces satisfies the formula $V - E + F = 2$.

He was not able to prove the formula but demonstrated its correctness on some examples. Later year he gave an incorrect proof. In 1796, Legendre proves it correctly for all convex polyhedra, later Cauchy in 1822. This history is mentioned and Cauchy's proof

¹Except Descartes according to a century-later revealed manuscript [Ri19], pp. 81-86.

²Richeson's book *Euler's Gem: The Polyhedron Formula and The Birth of Topology*, [Ri19] is a expository/historical book fully dedicated to this.

is amended in [BrTh21].

After Möbius', Listing's, Riemann's works in 1820-1860, Poincaré has generalized the formula to orientable surfaces of *genus* g and to higher dimensional manifolds, by Betti numbers, in his foundational paper *Analysis Situs* in 1895. Most importantly, he asserted that the then-extended constant, the Euler characteristic χ , was a topological invariant.

Later in the 20th century, Veblen first rigorously proved the topological invariance of the Euler characteristic for polyhedra. J.W. Alexander rigorously introduced simplicial homology, defined the characteristic in terms of homology and gave a homological proof ([A115]). Through Noether, an algebraic presentation was given and later Eilenberg and Steenrod extended the invariance to homotopy ([Ri19], pp. 253-264; [EiSt45]).

Here, we prematurely present the modern combinatorial definition of the Euler characteristic of a CW complex (thus of a Δ -complex, or of a simplicial complex):

Definition 1.1 (Euler characteristic, [Ha02], p. 146). Let X be a finite CW-complex and let a_i be the number of i -cells in X . Then the Euler characteristic of X , $\chi(X)$, is defined as follows:

$$\chi(X) = \sum_i (-1)^i a_i$$

1.2 Motivating The Project

The main goal of this project is to prove the following:

Theorem 1.2. Given a finite CW-complex X , $\chi(X)$ is invariant under homotopy. In particular, it is independent of the complex on the space.

Before motivating the project, we can observe this on some restricted cases. If we momentarily return to the Platonic solids, particularly the cube, we see this formula gives the same number for homeomorphic complexes. Here we restrict the formula to polyhedra by an alternating sum of number of simplices.

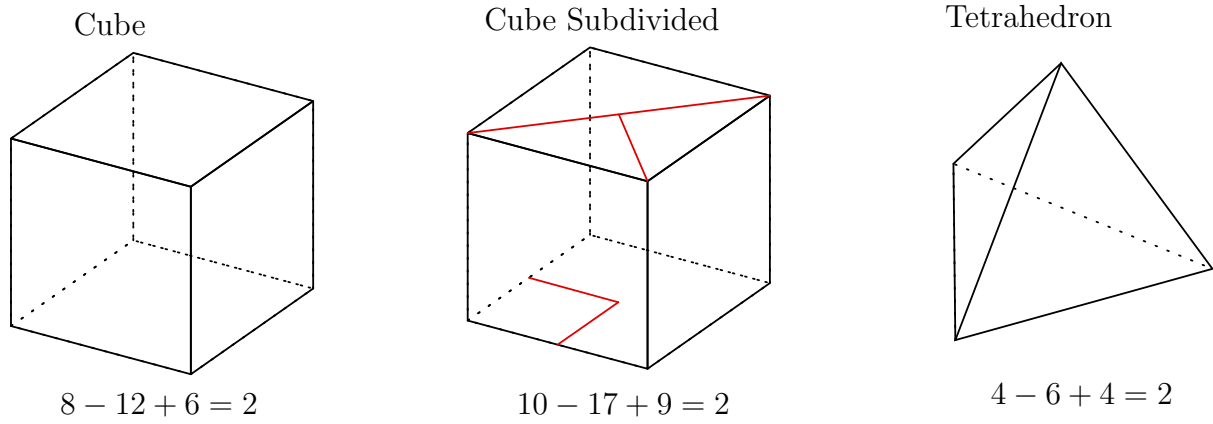


Figure 2: Demonstration for homeomorphic complexes

We can also try to visualise the formula for a homotopy equivalent yet not homeomorphic complexes (See [Ha02], p. 3 for the details of this example). Here, the formula is restricted to Δ -complexes (see Definition A.16), again as the alternating sum of number of simplices.³

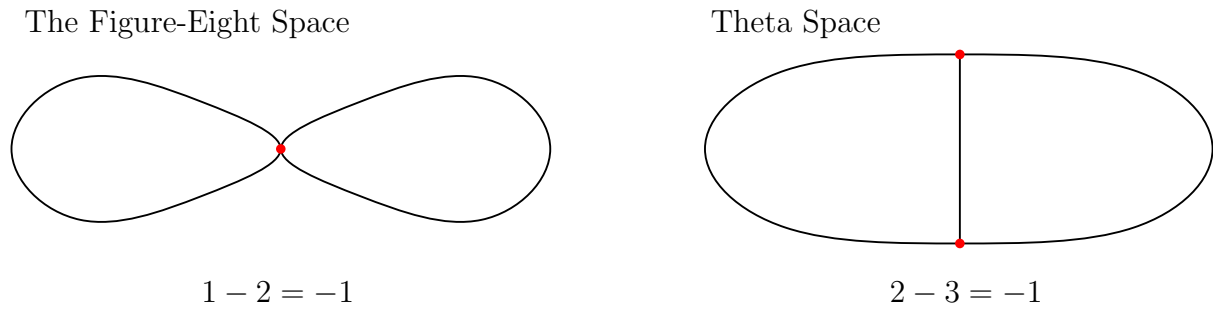


Figure 3: Demonstration for homotopy-equivalent (but not homeomorphic) complexes

The problem is that, in general, it is practically impossible to prove homotopy invariance of the Euler characteristic, let alone the topological invariance, using this definition. Even in simplicial complexes, since *Hauptvermutung* - roughly speaking, the theorem stating that two different complexes can be refined by subdivision to reach to a common complex - is refuted⁴, it is practically not possible to prove topological invariance. The case gets more ambiguous as one moves forward to CW complexes.

At this point, if one temporarily abandons this combinatorial/geometric approach and develop the theory of homology of spaces and complexes, then later a “homological” defi-

³See Example 4.3 for the figure-eight as part of the skeleta of the torus T .

⁴See Poincaré’s assumption of *Hauptvermutung* in [Ri19], p. 262.

inition of the Euler characteristic can be equated to the combinatorial, proving homotopy invariance and many other things. This is the main theorem of this project.

Theorem 1.3 (Euler-Poincaré, [Br97]). Let X be a finite CW-complex. Then we have:

$$\chi(X) = \sum_i (-1)^i \text{rank } H_i(X)$$

Therefore, **the roadmap** of the project is to introduce singular homology, prove main properties (including homotopy invariance), introduce CW-complexes and cellular homology, show their equivalence, and finally prove the theorem above.

All maps in this project are assumed to be continuous unless specified otherwise.

2 Singular Homology

Definition 2.1. Let $\mathbb{R}^\omega$ have the standard basis $\{e_0, e_1, \dots\}$. Then

$$\Delta_p := \left\{ x = \sum_{i=0}^p \lambda_i e_i \mid \sum_{i=0}^p \lambda_i = 1, 0 \leq \lambda_i \leq 1 \right\}$$

is called the **standard p -simplex** and λ_i are called the **barycentric coordinates**.

Definition 2.2. Let $v_0, \dots, v_n$ be points in $\mathbb{R}^N$. The map:

$$[v_0, \dots, v_n] : \Delta_n \rightarrow \mathbb{R}^N, \quad \sum_{i=0}^n \lambda_i e_i \mapsto \sum_{i=0}^n \lambda_i v_i$$

is called an **affine singular n -simplex**.

Remark 2.1. This canonical linear homomorphism takes the **ordering** of the vertices as a convention from v_0 to v_n , i.e. now any edge $[v_i, v_j]$ has **orientation** according to increasing subscripts.

Definition 2.3. Denote the hat above as omitting a vertex. The affine singular $(p-1)$ -simplex $[e_0, \dots, \hat{e}_i, \dots, e_p] : \Delta_{p-1} \rightarrow \Delta_p$ is called the **i th face map**, or face, denoted F_i^p .

Definition 2.4. Let X be a topological space. Then a map $\sigma_p : \Delta_p \rightarrow X$ is called a **singular p -simplex** of X . The free abelian group^a on the singular p -simplices $\{\sigma_p\}_\sigma$ is called the **singular p -chain group**, denoted $\Delta_p(X)$. An element of $\Delta_p(X)$ is called a **p -chain** and it is a formal sum $c = \sum_\sigma n_\sigma \sigma$ over all singular p -simplices with all but finite coefficients $n_\sigma \in \mathbb{Z}$ zero.

^acf. Definition A.2.

Definition 2.5. Let $\sigma : \Delta_p \rightarrow X$ be a singular p -simplex. The **i th face** of σ is $\sigma^{(i)} := \sigma \circ F_i^p$.

The **boundary** of σ is $\partial_p \sigma := \sum_{i=0}^p (-1)^i \sigma^{(i)}$. ∂_p extends uniquely as:

$$\begin{aligned} \partial_p : \Delta_p(X) &\rightarrow \Delta_{p-1}(X) \\ \partial_p\left(\sum_\sigma n_\sigma \sigma\right) &\mapsto \sum_\sigma n_\sigma \partial_p \sigma \end{aligned}$$

Remark 2.2. Not only denotational but also a useful remark is about the composition of face maps: $F_i^p(e_j) = e_j$ if $j < i$ and $F_i^p(e_j) = e_{j+1}$ if $j \geq i$. Then:

$$\begin{aligned} F_j^{p+1} \circ F_i^p &= [e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_p] \quad \text{for } j > 1 \\ F_j^{p+1} \circ F_i^p &= [e_0, \dots, \hat{e}_j, \dots, \hat{e}_{i+1}, \dots, e_p] \quad \text{for } j \leq i \end{aligned}$$

This is easily understood if the face maps are thought in terms of, informally speaking, *annexing a zero* and *mapping the omittance*.

Lemma 2.1. $\partial_p \partial_{p+1} = 0$ for $p \geq 0$.

Proof. It suffices to show on a singular p -simplex σ :

$$\begin{aligned} \partial_p \partial_{p+1} \sigma &= \partial_p \sum_{i=0}^{p+1} (-1)^i \underbrace{(\sigma \circ F_i^{p+1})}_{\text{a } p\text{-simplex}} = \partial_p \sum_{i=0}^{p+1} (-1)^i \partial_p \sum_{j=0}^p (-1)^j (\sigma \circ F_i^{p+1}) \circ F_j^p \\ &= \partial_p \sum_{i=0}^{p+1} \partial_p \sum_{j=0}^p (-1)^{i+j} \sigma \circ F_i^{p+1} \circ F_j^p \\ &= \sum_{0 \leq i < j \leq p} (-1)^{i+j} \sigma \circ F_i^{p+1} \circ F_j^p + \sum_{0 \leq j \leq i \leq p+1} (-1)^{i+j} \sigma \circ F_i^{p+1} \circ F_j^p \\ &= \sum_{0 \leq i < j \leq p} (-1)^{i+j} \sigma \circ F_i^{p+1} \circ F_j^p + \sum_{0 \leq j \leq i \leq p+1} (-1)^{i+j} \sigma \circ F_{j+1}^{p+1} \circ F_i^p \end{aligned}$$

The last equality follows from the previous remark as $F_i^{p+1} \circ F_j^p = F_{j+1}^{p+1} \circ F_i^p$. Observe that substituting $i + 1$ in i proves the lemma. $\square$

Definition 2.6. Conveniently, it is defined that $\Delta_p(X) = 0$ for $p < 0$, and $\partial_p = 0$ for $p \leq 0$. Therefore, the following composition is *always* zero:

$$\Delta_{p+1}(X) \xrightarrow{\partial_{p+1}} \Delta_p(X) \xrightarrow{\partial_p} \Delta_{p-1}(X)$$

Definition 2.7. We define *p-cycles* and *p-boundaries* as follows:

$$\begin{aligned} p\text{-cycles} &= \ker \partial_p = Z_p(X) \\ p\text{-boundaries} &= \operatorname{im} \partial_{p+1} = B_p(X) \end{aligned}$$

Definition 2.8. The *p*th singular homology group of a space X is defined as follows:

$$H_p(X) = Z_p(X) / B_p(X)$$

Remark 2.3. Given two chains c_1 and c_2 , they are said to be **homologous** if $c_1 - c_2 = \partial d$ for some $(p+1)$ -chain d , in other words, c_1 and c_2 are homologous if their formal difference is in $\operatorname{im} \partial_{p+1}$. It is denoted $c_1 \sim c_2$ and homology is an equivalence relation on cycles. Moreover, we have $H_p(X) = Z_p(X) / \sim$.

2.1 Additivity and Functoriality

Let $f : X \rightarrow Y$ be a map and for some $p \geq 0$, let $\sigma \in \Delta_p(X)$. Then $f \circ \sigma \in \Delta_p(Y)$. This extends uniquely to a homomorphism on chain groups:

$$f_{\#} : \Delta_p(X) \rightarrow \Delta_p(Y) \text{ by } f_{\#}(\sum_{\sigma} n_{\sigma} \sigma) = \sum_{\sigma} n_{\sigma} (f \circ \sigma)$$

Proposition 2.2. For a given map $f : X \rightarrow Y$ and given $p \geq 0$, the induced homomorphism on chain groups $f_{\#} : \Delta_*(X) \rightarrow \Delta_*(Y)$ satisfies $f_{\#} \circ \partial = \partial \circ f_{\#}$ (I.e. $f_{\#}$ is a chain map).

Proof. As it is sufficient to show the property on a generator, given $\sigma \in \Delta_p(X)$, observe

that:

$$\partial(f_{\#}(\sigma)) = \partial(f \circ \sigma) = \sum_{i=0}^p (-1)^i (f \circ \sigma) \circ F_i^p = \sum_{i=0}^p (-1)^i f \circ (\sigma \circ F_i^p) = f_{\#}(\partial(\sigma))$$

Hence $f_{\#} \circ \partial = \partial \circ f_{\#}$. □

Corollary 2.3. *A map $f : X \rightarrow Y$, induces a homomorphism on the homology groups $f_* : H_p(X) \rightarrow H_p(Y)$, with the additional properties that*

$$(id_X)_* = id_{H_p(X)} \quad \text{and} \quad (f \circ g)_* = f_* \circ g_*$$

Proof. Let $c \in \Delta_p(X)$ be a cycle, then because $f_{\#}$ is a chain map, $f_{\#}(c)$ is also a cycle in $\Delta_p(Y)$. For the same reason, a homologous pair is mapped to a homologous pair. Thus, f_* is a well-defined homomorphism. The additional properties follow from definition. □

Corollary 2.4. *If $f : X \rightarrow Y$ is a homeomorphism, then $f_* : H_p(X) \rightarrow H_p(Y)$ is an isomorphism. I.e. Homology groups are invariant under homeomorphisms.*

Theorem 2.5 (Additivity). Let X be a space where its path-components are given by $\{X_\alpha\}_{\alpha \in A}$. Then $H_p(X) = \bigoplus_\alpha H_p(X_\alpha)$.

Proof. The following proof is an expanded version of the proof sketch given in [Ha02], p. 109.

Let $i_\alpha : X_\alpha \hookrightarrow X$ be inclusion. Then assembling the induced maps $(i_\alpha)_\#^n$ leads to the isomorphism at the chain level:

$$\begin{aligned} \varphi_n &= \bigoplus_{\alpha \in A} (i_\alpha)_\#^n : \bigoplus_{\alpha \in A} \Delta_n(X_\alpha) \rightarrow \Delta_n(X) \\ &\sum_\alpha m_\alpha \sigma_\alpha \mapsto \sum_\alpha m_\alpha (i_\alpha)_\#(\sigma_\alpha) \end{aligned}$$

This is a well-defined homomorphism that is clearly injective. For surjection, observe that for a generator $\sigma : \Delta_n \rightarrow X$, the simplex Δ_n is path-connected and σ is continuous, therefore the image must lie in a single path-component, let X_β . Then $\sigma \in \Delta_n(X_\beta)$, thus $\varphi_n(\sigma) = \sigma$, i.e. surjection. Hence φ_n is an isomorphism of groups.

Since by Proposition 2.2, induced maps commute with the boundary homomorphism, considering the definition of φ_n , we also have:

$$\varphi_n \circ \bigoplus_{\alpha \in A} \partial_n^{X_\alpha} = \partial_n^X \circ \varphi_{n+1}$$

In other words, the boundary maps preserve the direct sum decomposition of $\Delta_n(X)$. Using this observation, we can conclude the following.

Let $a \in \Delta_n(X)$. Then:

$$\begin{aligned} a \in \ker \partial_{n-1}^X &\iff \partial_{n-1}^X(a) = 0 \\ &\iff (\varphi_{n-1} \circ \bigoplus_{\alpha \in A} \partial_{n-1}^{X_\alpha} \circ \varphi_n^{-1})(a) = 0 \\ &\iff \bigoplus_{\alpha \in A} \partial_{n-1}^{X_\alpha}(\varphi_n^{-1}(a)) = 0 \\ &\iff \varphi_n^{-1}(a) \in \ker \bigoplus_{\alpha \in A} \partial_{n-1}^{X_\alpha} \\ &\iff a \in \varphi_n(\ker \bigoplus_{\alpha \in A} \partial_{n-1}^{X_\alpha}) \end{aligned}$$

Hence $\ker(\partial_{n-1}^X) = \varphi_n(\ker \bigoplus_{\alpha \in A} \partial_{n-1}^{X_\alpha})$. Same argument can be applied to obtain $\text{im}(\partial_n^X) = \varphi_n(\text{im} \bigoplus_{\alpha \in A} \partial_n^{X_\alpha})$. Thus,

$$\ker(\partial_{n-1}^X) / \text{im}(\partial_n^X) = \varphi_n(\ker \bigoplus_{\alpha \in A} \partial_{n-1}^{X_\alpha}) / \varphi_n(\text{im} \bigoplus_{\alpha \in A} \partial_n^{X_\alpha}) \cong \ker(\bigoplus_{\alpha \in A} \partial_{n-1}^{X_\alpha}) / \text{im}(\bigoplus_{\alpha \in A} \partial_n^{X_\alpha})$$

In other words, $\ker \partial_{n-1}$ and $\text{im} \partial_n$ split as direct sums.

At last, giving us:

$$\begin{aligned} H_n(X) &= \ker(\partial_{n-1}^X) / \text{im}(\partial_n^X) \cong \ker(\bigoplus_{\alpha \in A} \partial_{n-1}^{X_\alpha}) / \text{im}(\bigoplus_{\alpha \in A} \partial_n^{X_\alpha}) \\ &\cong \bigoplus_{\alpha \in A} \ker(\partial_{n-1}^{X_\alpha}) / \bigoplus_{\alpha \in A} \text{im}(\partial_n^{X_\alpha}) \cong \bigoplus_{\alpha \in A} \ker(\partial_{n-1}^{X_\alpha}) / \text{im}(\partial_n^{X_\alpha}) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha) \end{aligned}$$

□

Remark 2.4. One important consequence of the proposition before is that for homology, it is sufficient to study path-connected spaces.

2.2 Dimension Property

Computing the homology groups of spaces with singular homology is done in a round-about way using other tools, mainly due to the non-strict definition of it which results in huge groups. The one-point space is about the only space that we can directly compute. The following result is denoted as the dimension property of singular homology.

Example 2.1 (Dimension). Let Q be the one-point space. We have unique maps $\sigma : \Delta_p \rightarrow Q$ for all $p \geq 0$. Also, a p -simplex σ has $p + 1$ faces, all of which are the same. Though in $\partial_p \sigma$ they are presented with alternating signs. Thus $\partial_p \sigma = 0$ if p is odd. If $p > 0$ is even, $\partial_p \sigma^p = +\sigma^{p-1}$, so $\text{im } \partial_p = \langle +\sigma^{p-1} \rangle \cong \mathbb{Z}$ but $\ker \partial_p = 0$ as ∂_p is an isomorphism in this case.

If $p = 0$, $\partial_0 = 0$ by definition and $\partial_1 \sigma = 0$, thus, put altogether:

$$H_n(Q) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ 0 & \text{if } p \neq 0 \end{cases}$$

2.3 Exactness and Homological Algebra

Here, we introduce some purely algebraic notions that we use to derive some consequences on homology. All of these notions are used extensively hereafter.

Definition 2.9. A **graded group** $\{C_i\}$ is a collection of abelian groups indexed by $\mathbb{Z}$.

Definition 2.10. A **chain complex** $C_* = (\{C_i\}, \partial)$ is a graded group $\{C_i\}$ with a sequence of homomorphisms $\partial : C_i \rightarrow C_{i-1}$ s.t. $\partial^2 = 0$. ∂ is called a **boundary operator** or **differential**.

Example 2.2. Recalling the notation we have used in Subsection 2.1, Additivity and Functoriality, it is seen that with $C_i = \Delta_i(X)$ for $i \geq 0$ and $C_i = 0$ for $i < 0$; with also the boundary homomorphism ∂ , we have the singular chain complex, shortly denoted $\Delta_*(X)$.

At this point, we are able to define homology completely algebraically as follows.

Definition 2.11. If $C_* = (\{C_i\}, \partial)$ is a chain complex, then we define the **homology (of the chain complex C_*)** as the following graded group:

$$H_i(C_*) = \frac{\ker \partial : C_i \rightarrow C_{i-1}}{\operatorname{im} \partial : C_{i+1} \rightarrow C_i}$$

Example 2.3. As the example, we can now see the singular homology groups as the graded group consisting of the (algebraic) homology groups of the singular homology complex:

$$H_i(X) = H_i(\Delta_*(X))$$

We can define maps between chain complexes, with an appropriate commuting condition, as follows.

Definition 2.12. Let A_* and B_* be two chain complexes. Then a **chain map** $f : A_* \rightarrow B_*$ is a collection of homomorphisms $f : A_i \rightarrow B_i$ such that:

$$f \circ \partial = \partial \circ f$$

I.e. a chain map is the following commuting *ladder*:

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & A_{i+1} & \xrightarrow{\partial} & A_i & \xrightarrow{\partial} & A_{i-1} & \xrightarrow{\partial} & \dots \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ \dots & \xrightarrow{\partial} & B_{i+1} & \xrightarrow{\partial} & B_i & \xrightarrow{\partial} & B_{i-1} & \xrightarrow{\partial} & \dots \end{array}$$

Proposition 2.6. A chain map $f : A_* \rightarrow B_*$ induces a homomorphism of graded homology groups $f_* : H_*(A_*) \rightarrow H_*(B_*)$ by $f_*[a] = [f(a)]$, such that $(f \circ g)_* = f_* \circ g_*$ (given another chain map g) and $1_* = 1$.

Proof. This follows from functoriality of induced homomorphisms of each f applied to the whole sequence. □

Definition 2.13. A sequence of groups

$$A \xrightarrow{i} B \xrightarrow{j} C$$

is called **exact** if $\operatorname{im}(i) = \ker(j)$.

From this point on, some facts about exact sequences, particularly, how they are used in establishing isomorphisms, splitting of exact sequences, and so on, are taken for granted.

Theorem 2.7. A short exact sequence of chain complexes and chain maps

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

induces a long exact sequence

$$\dots \xrightarrow{\partial_*} H_p(A_*) \xrightarrow{i_*} H_p(B_*) \xrightarrow{j_*} H_p(C_*) \xrightarrow{\partial_*} H_{p-1}(A_*) \xrightarrow{i_*} \dots$$

where $\partial_* \llbracket c \rrbracket := \llbracket i^{-1} \circ \partial \circ j^{-1}(c) \rrbracket$ and it is called the **connecting homomorphism**.

Proof. The proof is done by *diagram chasing* on the diagram below, see [Br97] for the technical details:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{p+1} & \xrightarrow{i} & B_{p+1} & \xrightarrow{j} & C_{p+1} & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & A_p & \xrightarrow{i} & B_p & \xrightarrow{j} & C_p & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & A_{p-1} & \xrightarrow{i} & B_{p-1} & \xrightarrow{j} & C_{p-1} & \longrightarrow & 0 \end{array}$$

Here, diagram chasing refers to carrying elements on this diagram by the map ∂_* to see its well-definition, homomorphism; and exactness for the homology sequence. $\square$

To be able to continue to the theory singular homology in its most general version, we must *extend* our definition of homology to include pairs of spaces (X, A) where $A \subseteq X$.

Definition 2.14 (Relative Homology). Let (X, A) be a pair of spaces s.t. $A \subseteq X$. We have the canonical inclusions $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$. Then, recalling the chain groups as the free groups generated by simplices, $\Delta_i(A)$ is a subgroup of $\Delta_i(X)$ and i is a chain map. Define the chain group of the pair as $\Delta_i(X, A) = \frac{\Delta_i(X)}{\Delta_i(A)}$. Then the following is an exact sequence:

$$0 \longrightarrow \Delta_*(A) \xrightarrow{i_*} \Delta_*(X) \xrightarrow{j_*} \Delta_*(X, A) \longrightarrow 0$$

The **relative homology** of (X, A) is defined as:

$$H_p(X, A) = H_p(\Delta_*(X, A))$$

Theorem 2.8 (Exactness). The relative (singular chain complex) sequence in the previous definition induces the exact homology sequence of the pair (X, A) :

$$\dots \xrightarrow{\partial_*} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{i_*} \dots$$

Proof. Apply Theorem 2.7, to Definition 2.14. □

There is one important technical thing we have to prove about ∂_* , to utilize it in establishing the commutativity of some diagrams later.

Lemma 2.9. *The connecting homomorphism ∂_* is a natural transformation (see Definition A.13).*

Proof. ∂_* being a natural transformation in this case means that given a map $f : (X, A) \rightarrow (Y, B)$, the following square commutes:

$$\begin{array}{ccc} H_p(X, A) & \xrightarrow{\partial_*} & H_{p-1}(A) \\ f_* \downarrow & & \downarrow f_* \\ H_p(Y, B) & \xrightarrow{\partial_*} & H_{p-1}(B) \end{array}$$

Naturality follows from the definition of ∂_* . We refer the reader to p. 14 of [Ro12] for how. □

We shall now develop the algebraic theory of homology further to utilize it hereafter. We give the definitions of the homology with coefficients, reduced homology; and we

present a very useful lemma, called the 5-lemma.

Definition 2.15 (Homology with Coefficients). Given an abelian group G , the tensor product $\Delta_*(X) \otimes G$ is a chain complex with the differential $\partial \otimes 1$. Define the **homology groups with coefficients in G** by

$$H_p(X; G) = H_p(\Delta_* \otimes G)$$

We can also define **relative homology with coefficients in G** as

$$H_p(X, A; G) = H_p(\Delta_*(X, A) \otimes G).$$

Proposition 2.10. *Homology with coefficients serve as a full homology theory satisfying the Eilenberg-Steenrod axioms.*

Proof. We do not go into details of tensoring and take this for granted. See p. 181 of [Br97], and for the algebra behind exactness, see p. 399 of [DuFo04]. $\square$

Definition 2.16 (Reduced homology). Let C_* be the chain complex defined by $C_i = \Delta_i(X)$ for $i \geq 0$, $C_{-1} = \mathbb{Z}$, and $C_i = 0$ for $i < -1$, with the differential $C_0 \rightarrow C_{-1}$ is the augmentation map $\varepsilon : \Delta_0(X) \rightarrow \mathbb{Z}$. Define the homology of this complex as the **reduced homology**, denoted $\tilde{H}_*(X)$.

Remark 2.5. Reduced homology differs from $H_*(X)$ only in degree zero. For an algebraic definition of reduced homology and to better see how it differs from absolute homology, see Definition 3.3, and Theorem 3.3.

Theorem 2.11 (5-lemma). Let f_1, f_2, f_4, f_5 be isomorphisms in the following commutative diagram with exact rows, then f_3 is also an isomorphism:

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & f_4 \downarrow & & f_5 \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

Proof. Straightforward diagram-chase. Injectivity and surjectivity can be shown directly. $\square$

2.4 Homotopy Invariance and Excision

Proving for the homotopy invariance and excision properties of singular homology is most technical in its machineries. Therefore, we concisely present homotopy invariance and refer the reader to pp. 223-228 of [Br97] for excision.

Definition 2.17. Let $\varphi, \psi : A_* \rightarrow B_*$ be chain maps between chain complexes A_*, B_* . φ and ψ are **chain homotopic**, denoted $\varphi \simeq \psi$, if there exists a sequence of homomorphisms $D : A_i \rightarrow B_{i+1}$ s.t. $\partial D + D\partial = \varphi - \psi$.

Proposition 2.12. If $\varphi \simeq \psi : A_* \rightarrow B_*$, then $\varphi_* = \psi_* : H_*(A_*) \rightarrow H_*(B_*)$.

Proof. Follows from chain homotopy definition. □

Proposition 2.13. Chain homotopy is an equivalence relation on the set of chain maps between two chain complexes.

Proof. Again, follows from chain homotopy definition. □

Definition 2.18. A chain map $\varphi : A_* \rightarrow B_*$ is called a **chain equivalence** if there is a chain map $\psi : B_* \rightarrow A_*$ such that $\varphi\psi \simeq 1_{B_*}$ and $\psi\varphi \simeq 1_{A_*}$.

The following is a technical theorem that prepares the construction of cross product of homology groups.

Theorem 2.14. For all $p, q \in \mathbb{Z}$, there exist bilinear maps $\delta : \Delta_p(X) \times \Delta_q(Y) \rightarrow \Delta_{p+q}(X \times Y)$ s.t.:

- (1) For $x \in X$, $y \in Y$, a p -simplex σ in Y , a q -simplex τ in X , $x \times \sigma$ is the q -simplex of $X \times Y$ given by $w \mapsto (x, \sigma(w))$ and symmetrically for $\tau \times y$,
- (2) (Naturality) if $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two maps where $\langle f, g \rangle : X \times Y \rightarrow X' \times Y'$, then $\langle f, g \rangle_{\#}(a \times b) = f_{\#}(a) \times g_{\#}(b)$,
- (3) (Boundary formula) $\partial(a \times b) = \partial a \times b + (-1)^{\deg a} \times \partial b$.

Proof. See p. 222 of [Br97]. □

To be able to operate on pairs, we need to define products of pairs.

Definition 2.19. Let (X, A) and (Y, B) be two pairs of spaces, then $(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$.

Theorem 2.15. The cross product $\Delta_p(X) \times \Delta_q(Y) \rightarrow \Delta_{p+q}(X \times Y)$ induces a bilinear map $\delta : H_p(X, A) \times H_q(Y, B) \rightarrow H_{p+q}((X, A) \times (Y, B))$ given by $\llbracket a \rrbracket \times \llbracket b \rrbracket = \llbracket a \times b \rrbracket$.

Proof. See p. 222 of [Br97] for how the previous theorem is applied here. $\square$

Theorem 2.16. If $f_0 \simeq f_1 : (X, A) \rightarrow (Y, B)$, then $(f_0)_\# \simeq (f_1)_\# : \Delta_*(X, A) \rightarrow \Delta_*(Y, B)$, thus $(f_0)_* = (f_1)_* : H_*(X, A) \rightarrow H_*(Y, B)$.

Proof. See p. 223 of [Br97]. $\square$

Corollary 2.17. Singular homology satisfies the homotopy invariance axiom/property.

3 Eilenberg-Steenrod Axioms and The Axiomatic Perspective

After various homology theories by various mathematicians, Eilenberg and Steenrod had formalized homology theories under a few axioms in their 1945 paper, *Axiomatic approach to homology theory* ([EiSt45]).

In this section, the axioms are presented. It will be demonstrated that we have already proven singular homology as satisfying the axioms (as we denoted so far, *properties*). Then algebraic consequences will be derived from the axiomatic perspective to construct cellular homology later. All of derived properties can specifically be considered as properties of singular homology.

Definition 3.1. A **homology theory** is a functor $H : \mathbf{Top}^2 \rightarrow \mathbf{Ab}$ that assigns:

$$(X, A) \mapsto \{H_p(X, A)\}$$

$$f : (X, A) \rightarrow (Y, B) \mapsto f_* : H_p(X, A) \rightarrow H_p(Y, B)$$

in addition with a natural transformation of functors $\partial_* : H_p(X, A) \rightarrow H_{p-1}(A)$, called the **connecting homomorphism**, such that the **Eilenberg-Steenrod axioms** are satisfied. Here $\mathbf{Top}^2$ is the category of all pairs of topological spaces (X, A) with $A \subseteq X$.

Definition 3.2. The following are the Eilenberg-Steenrod axioms of a homology theory:

(1) (Homotopy invariance axiom.)

$$f \simeq g : (X, A) \rightarrow (Y, B) \implies f_* = g_* : H_*(X, A) \rightarrow H_*(Y, B)$$

(2) (Exactness axiom.) Given the inclusions $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$, the following sequence is exact:

$$\dots \xrightarrow{\partial_*} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{i_*} \dots$$

(3) (Excision axiom.) Given a pair (X, A) and an open set $U \subset X$ such that $\overline{U} \subset \text{int}(A)$, the inclusion $k : (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism:

$$k_* : H_*(X - U, A - U) \xrightarrow{\cong} H_*(X, A)$$

(4) (Dimension axiom.) For a one-point space P , $H_i(P) = 0 \quad \forall i \neq 0$.

(5) (Additivity axiom.) For a topological sum $X = \bigoplus_{\alpha} X_{\alpha}$, the following homomorphism is an isomorphism:

$$\bigoplus (i_{\alpha})_* : \bigoplus_{\alpha} H_n(X_{\alpha}) \rightarrow H_n(X)$$

where $i_{\alpha} : X_{\alpha} \hookrightarrow X$ is inclusion.

Theorem 3.1. Singular homology is a homology theory.

Proof. Dimension axiom is proven in Example 2.1. Additivity axiom is proven in Theorem 2.5. Exactness is proven in Theorem 2.8. For homotopy invariance and excision, see Subsection 2.4. For the naturality of ∂_* , see Lemma 2.9. $\square$

3.1 Immediate Results of A Homology Theory Satisfying E&S

In this subsection, we proceed into deriving some results concerning a homology theory. Despite this generality, all of these results can and should be thought to be particular to singular homology.

Proposition 3.2. *Given $(X, A) \simeq (Y, B)$, we have $H_*(X, A) \cong H_*(Y, B)$.*

Proof. Follows from the homotopy axiom and functoriality of H . □

Definition 3.3. Reduced homology can be defined completely algebraically as $\tilde{H}_0(X) = \ker \varepsilon_*$ where $\varepsilon : X \rightarrow P$ and $\varepsilon_* : H_*(X) \rightarrow H_*(P)$ and P the one-point space.

Given the fact that ε_* is onto as $\varepsilon \circ i = 1$, where $i : P \rightarrow X$ any map, the definition of the (zeroth) reduced homology group is equivalent to the exactness of the sequence:

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \xrightarrow{\varepsilon_*} H_0(P) = G \longrightarrow 0$$

This sequence splits, not naturally, via i_* , hence $H_0(X) \cong \tilde{H}_0(X) \oplus G$. This establishes the relationship between the absolute homology group and the reduced homology group.

For $p \neq 0$ we set $\tilde{H}_0(X) = H_0(X)$, and for all p and for $A \neq \emptyset$, we set $\tilde{H}_p(X, A) = H_p(X, A)$.

Theorem 3.3. Reduced homology preserves the axioms.

Proof. Given a homology theory, it is immediately seen that its reduced version satisfies homotopy, excision, dimension, and additivity axioms as those results do not depend on $\tilde{H}_0(X)$. For the exactness axiom, it suffices to prove exactness at $\tilde{H}_1(X, A)$, $\tilde{H}_0(A)$, $\tilde{H}_0(X)$, and at $\tilde{H}_0(X, A)$ at the following segment:

$$\dots \xrightarrow{j_*} \tilde{H}_1(X, A) \xrightarrow{\partial_*} \tilde{H}_0(A) \xrightarrow{i_*} \tilde{H}_0(X) \xrightarrow{j_*} \tilde{H}_0(X, A) \xrightarrow{\partial_*} 0$$

The map $(X, A) \rightarrow (P, P)$, through naturality of ∂_* , induces the following commutative

diagram:

$$\begin{array}{ccccccc}
 & & \tilde{H}_0(A) & \longrightarrow & \tilde{H}_0(X) & & \\
 & \nearrow & \downarrow & & \downarrow & \searrow & \\
 H_1(X, A) & \longrightarrow & H_0(A) & \longrightarrow & H_0(X) & \longrightarrow & H_0(X, A) \longrightarrow H_{-1}(A) = 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 = H_1(P, P) & \longrightarrow & H_0(P) & \longrightarrow & H_0(P) & \longrightarrow & H_0(P, P) = 0
 \end{array}$$

A diagram chase on this diagram proves the exactness axiom. $\square$

Theorem 3.4. If X is contractible, then $\tilde{H}_*(X) = 0$.

Proof. Apply Proposition 3.2 to an arbitrary space X and to the single-point space P . $\square$

Theorem 3.5. Denote the n -dimensional closed upper hemisphere of $\mathbb{S}^n$ by $\mathbb{D}_+^n$, and the lower as $\mathbb{D}_-^n$. Then for $n \geq 0$, we have the following statements (S_n) , (D_n) , and (R_n) of isomorphisms:

$$\begin{aligned}
 (S_n) : \tilde{H}_i(\mathbb{S}^n) &\cong \begin{cases} G, & \text{if } i = n, \\ 0, & \text{if } i \neq n. \end{cases} \\
 (D_n) : H_i(\mathbb{D}^n, \mathbb{S}^{n-1}) &\cong \begin{cases} G, & \text{if } i = n, \\ 0, & \text{if } i \neq n. \end{cases} \\
 (R_n) : H_i(\mathbb{S}^n, \mathbb{D}_+^n) &\cong \begin{cases} G, & \text{if } i = n, \\ 0, & \text{if } i \neq n. \end{cases}
 \end{aligned}$$

Proof. For the base case of an inductive proof, (R_0) follows from $H_i(\mathbb{S}^0, \mathbb{D}_+^0) \cong H_i(P) \cong G$, where the isomorphisms follows from excision and dimension axioms.

For $(R_n) \iff (S_n)$, observe that the map $\mathbb{D}_+^n \hookrightarrow \mathbb{S}^n$, induces the exact reduced homology sequence:

$$0 = \tilde{H}_i(\mathbb{D}_+^n) \longrightarrow \tilde{H}_i(\mathbb{S}^n) \longrightarrow H_i(\mathbb{S}^n, \mathbb{D}_+^n) \longrightarrow \tilde{H}_{i-1}(\mathbb{D}_+^n) = 0$$

where the end 0's are from contractibility, following Theorem 3.4, thus isomorphism of homology groups.

$(R_n) \iff (D_n)$ follows from $H_i(\mathbb{S}^n, \mathbb{D}_+^n) \cong H_i(\mathbb{S}^n - U, \mathbb{D}_+^n - U) \cong H_i(\mathbb{D}_-^n, \mathbb{S}^{n-1}) \cong H_i(\mathbb{D}^n, \mathbb{S}^{n-1})$ where the first two isomorphisms are from the excision axiom and then

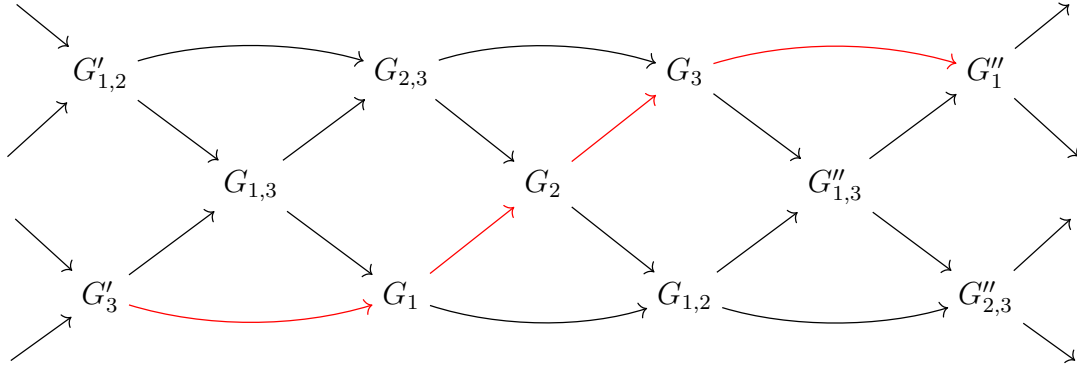
from Proposition 3.2, here U is a small enough subset on the north pole.

For $(S_{n-1}) \iff (D_n)$, follows from the exact sequence:

$$0 = \tilde{H}_i(\mathbb{D}^n) \longrightarrow H_i(\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow \tilde{H}_{i-1}(\mathbb{S}^{n-1}) \longrightarrow \tilde{H}_{i-1}(\mathbb{D}^n) = 0$$

where, again, the end zero's follow from contractibility. Therefore, by mathematical induction, we have $(D_0) \iff (R_0) \iff (S_0) \iff \dots \implies (D_n) \implies (R_n) \implies (S_n) \implies \dots$ $\square$

Lemma 3.6 ([Wa66] from p. 189 of [Br97]). *Consider the following commutative braid diagram, where the term braid denotes sequences:*



If the three braids (not counting the red braid with single subscripts) are exact and the single subscript braid is of order two, then it is also exact.

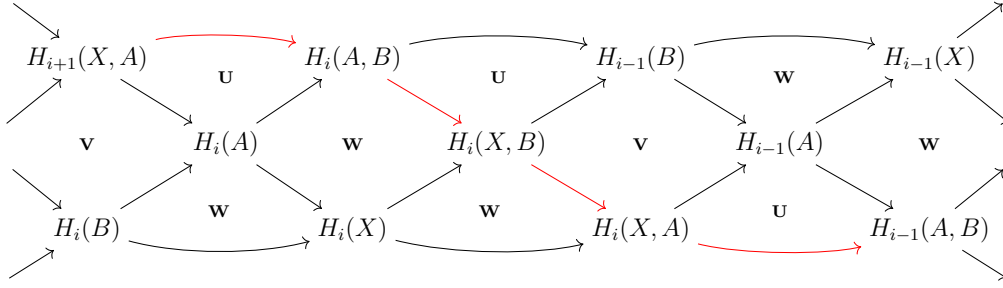
Proof. The proof is diagram chasing, from [Br97]. $\square$

Theorem 3.7 (Exact homology sequence of a triple (X, A, B)). Let $B \subseteq A \subseteq X$ and define $\partial_{**} : H_i(X, A) \rightarrow H_{i-1}(A, B)$ to be the composition $i_* \circ \partial_*$ where $\partial_* : H_i(X, A) \rightarrow H_{i-1}(A)$ is the connecting homomorphism of a tuple (X, A) and $i_* : H_{i-1}(A) \rightarrow H_{i-1}(A, B)$ the induced map by inclusion $A \hookrightarrow (A, B)$. Then the following sequence is exact, where other maps come from inclusions:

$$\dots \xrightarrow{\partial_{**}} H_p(A, B) \xrightarrow{i_*} H_p(X, B) \xrightarrow{j_*} H_p(X, A) \xrightarrow{\partial_{**}} H_{p-1}(A, B) \xrightarrow{i_*} \dots$$

Proof. Interweaving the three exact sequences of pairs - $(X, A), (X, B), (A, B)$ - at their intersections, we obtain the following (interlocking) braid diagram containing the fourth

sequence (shown in red) that we desire to prove its exactness of:



Note that this diagram commutes as the regions **W** commute due to being induced solely by inclusions, **U**'s commute by definition of ∂_{**} and **V**'s commute by the naturality of ∂_* . By the commutativity, the fourth sequence of the triple (X, A, B) is seen to be of order two to *except* for the composition $H_i(A, B) \rightarrow H_i(X, B) \rightarrow H_i(X, A)$. But, as this factors through $H_i(A, A)$, which is 0, entire braid is of order two. Thus, by the previous lemma (3.6) we have the desired exactness. $\square$

The following lemma is an adaptation of the definition of naturality to the connecting homomorphism of a triple.

Proposition 3.8. *The connecting homomorphism ∂_{**} is a natural transformation.*

Proof. Given two triples (X, A, B) and (Y, C, D) , and a chain map f , the statement of the proposition is equivalent to the outer rectangle in the following diagram to be commutative:

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\partial_*} & H_{n-1}(A) & \xrightarrow{i_*} & H_{n-1}(A, B) \\ f_* \downarrow & & (f|_A)_* \downarrow & & (f|_A)_* \downarrow \\ H_n(Y, C) & \xrightarrow{\partial_*} & H_{n-1}(C) & \xrightarrow{i_*} & H_{n-1}(C, D) \end{array}$$

The left square commutes by the naturality of ∂_* and the right square commutes by the functoriality of homology. $\square$

3.2 Degrees of Maps $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$

In order to introduce cellular homology, the notion of a degree of a map needs to be understood and computed. The following are some results on degrees of maps $\mathbb{S}^n \rightarrow \mathbb{S}^n$ for some fixed homology theory. We start by defining the degree.

Definition 3.4. If $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is a map, $\deg(f)$ is defined to be the integer such that $f_*(a) = \deg(f) \cdot a$ for all $a \in \tilde{H}_n(\mathbb{S}^n; \mathbb{Z}) \cong \mathbb{Z}$.

Functoriality of H directly implies the following.

Proposition 3.9. If $f, g : \mathbb{S}^n \rightarrow \mathbb{S}^n$ are two maps, then $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

The following proposition on the degree of the sign-changing map is used extensively to prove later results in this subsection.

Proposition 3.10. For $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with coordinates $x_0, \dots, x_n$, let $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be given by $f(x_0, x_1, \dots, x_n) = (-x_0, x_1, \dots, x_n)$. Then $\deg(f) = -1$.

Proof. The proof is by induction on n .

For $n = 0$, we have $\mathbb{S}^0 \subset \mathbb{R}^1$, $\mathbb{S}^0 = \{x, x'\}$ where $x = 1$ and $x' = -1$. Then $f(x) = -1 = x'$ and $f(x') = 1 = x$. Since, obviously we have the topological sum $\mathbb{S}^0 = \{x\} + \{x'\}$, by the additivity property, we have the isomorphism $H_0(\{x\}) \oplus H_0(\{x'\}) \rightarrow H_0(\mathbb{S}^0)$. Through the unique maps between homology groups of one-point spaces P , we have $H_0(\mathbb{S}^0) \cong H_0(P) \oplus H_0(P)$. By the map $\varepsilon : \mathbb{S}^0 \rightarrow P$ and the isomorphism above, we have $\varepsilon_* : H_0(\mathbb{S}^0) = H_0(P) \oplus H_0(P) \rightarrow H_0(P)$ given by $(a, b) \mapsto a + b$. Thus, by the $\ker \varepsilon_*$ definition of the reduced homology and this isomorphism for $H_0(\mathbb{S}^0)$, we have the following explicit description for $\tilde{H}_0(\mathbb{S}^0)$, thus also for f_* :

$$\tilde{H}_0(\mathbb{S}^0) = \{(a, -a) \in H_0(P) \oplus H_0(P) \cong H_0(\mathbb{S}^0)\}.$$

Hence, $f_*(a, -a) = (-a, a) = -1 \cdot (a, -a) \implies \deg(f) = -1$.

Suppose now that the proposition holds for $\mathbb{S}^k$ for $k < n$. We have the following commutative diagram:

$$\begin{array}{ccccccc} \tilde{H}_n(\mathbb{S}^n) & \xrightarrow{\cong_1} & H_n(\mathbb{S}^n, \mathbb{D}_+^n) & \xleftarrow{\cong_2} & H_n(\mathbb{D}_-^n, \mathbb{S}^{n-1}) & \xrightarrow{\cong_3} & \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ \tilde{H}_n(\mathbb{S}^n) & \xrightarrow{\cong_1} & H_n(\mathbb{S}^n, \mathbb{D}_+^n) & \xleftarrow{\cong_2} & H_n(\mathbb{D}_-^n, \mathbb{S}^{n-1}) & \xrightarrow{\cong_3} & \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \end{array}$$

All the isomorphisms follow from the Proof of Theorem 3.5. The isomorphisms $\cong_1$ come from $(S_n) \iff (D_n)$, $\cong_2$ from $(D_n) \iff (R_n)$, and $\cong_3$ from $(D_n) \iff (S_{n-1})$.

Thus, like one domino toppling another, the leftmost f_* has degree -1 and by induction, we have the proposition. $\square$

Corollary 3.11. *The antipodal map $(-1) : \mathbb{S}^n \rightarrow \mathbb{S}^n$ has degree $(-1)^{n+1}$.*

Proof. By composing different sign-changing maps, Propositions 3.10 and 3.9 give us the result. $\square$

Corollary 3.12. *For even n and for the antipodal map (-1) on $\mathbb{S}^n$, we have $(-1) \neq 1_{\mathbb{S}^n}$.*

Proof. Through homotopy-invariance of degrees. $\square$

Proposition 3.13. *Let A be a $n \times n$ nonsingular matrix in $\mathbb{R}$. As a map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, it can be extended to a map $\tilde{T} : \mathbb{S}^n \rightarrow \mathbb{S}^n$, by adding the point at $+\infty$. Moreover, the extended map has $\deg(\tilde{T}) = \text{sgn}(|A|)$ where sgn denotes the sign of the determinant $|A|$.*

Proof. By the stereographic projection, we can show that $\mathbb{R}^n \cong \mathbb{S}^n - \{N\}$. So, as $\mathbb{S}^n$ is compact, it is the one-point compactification of $\mathbb{R}^n$. Since T is nonsingular, it is a homeomorphism on $\mathbb{R}^n$, continuity at $+\infty$ follows directly as limits are preserved. Thus, such extension $\tilde{T}$ exists.

By Proposition 3.9, for nonsingular matrices, it is sufficient to prove the proposition on three kinds of elementary matrices.

The scalar-multiplication matrix which is diagonal with one term on the diagonal different from 1 and 0 is homotopic to the identity matrix, or to the matrix differing from identity by the sign of one entry on the diagonal. The homotopy is through such similar matrices.

The row-addition matrix which differs from identity by having one off-diagonal entry, is homotopic to the identity.

The row-interchange matrix is topologically a reflection through the hyperplane (of equal coordinates $\{x_i = x_j\}$ and of codimension-one). This reflection, homotopically by a rotation, is the standard reflection map in Proposition 3.10. Other cases as well follow immediately from 3.10. $\square$

Some very basic definitions related to manifolds; charts, atlases, differentiability, orientations, are directly assumed and not introduced in this project paper as we only have maps on $\mathbb{S}^n$. One definition we do need to give is of a regular value.

Definition 3.5 ([Le13], p. 105). Given a smooth map $\Phi : M \rightarrow N$ of smooth manifolds M, N ; a point $p \in M$ is said to be a **regular point/value** of Φ if the differential $d\Phi_p : T_p M \rightarrow T_{\Phi(p)} N$ is surjective.

Proposition 3.14. *Let $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a differentiable and let the north pole p be a regular value such that $\exists! q \in f^{-1}(p)$. Then $\deg(f) = \text{sgn}|J_f(q)|$, where $J_f(q)$ is the Jacobian of f at q and it is computed from coordinate systems at p and q that differ by a rotation of $\mathbb{S}^n$.*

Proof. The proof follows, roughly, a 2-step procedure: (1) “simplify” the map f as much as possible while tracking the degree, (2) construct a homotopy from the simplified map to the identity map.

The first goal is to rotate the domain such that the rotation maps q onto p . A rotation of $\mathbb{S}^n$ is an action of $SO(n+1)$. Since $SO(n+1)$ is path-connected, any rotation is homotopic to the identity, which has degree 1. The desired rotation exists, in the simplest form as rotation in the plane spanned by p and q . Thus, we can assume f to be a map such that $p \mapsto p$ uniquely.

We know df_p exists as f is differentiable and its matrix A is nonsingular as p is a regular value. Then, considered as a linear map on the chart $\mathbb{R}^n, \mathcal{L}$, of the matrix A ; we can take $\mathcal{L}^{-1}$ on $\mathbb{R}^n$ and using Proposition 3.13, we can extend it to $\tilde{\mathcal{L}}^{-1}$, to have:

$$\tilde{f} := \tilde{\mathcal{L}}^{-1} \circ f \implies d\tilde{f}_p = d(\tilde{\mathcal{L}}^{-1} \circ f)_p = d\tilde{\mathcal{L}}_p^{-1} \circ df_p = A^{-1}A = I.$$

From the last matrix multiplication, we see that if the new map has degree 1, the proposition will be proven whether df_p has degree +1 or -1. Suppose now $f = \tilde{f}$.

The differentiability is preserved during the simplification, we can thus apply Taylor’s Theorem for f about p in some local coordinates $x = (x_1, \dots, x_n)$, which simplifies as:

$$f(x) = f(p \equiv 0) + \underbrace{df_{p \equiv 0}}_{=I} \cdot x + \underbrace{g(x)}_{\text{higher order terms}} = x + g(x)$$

with the conditions $\|g(x)\| < \frac{1}{2}\|f(x)\|$ for $0 < \|x\| \leq 2\varepsilon$ for some $\varepsilon > 0$. The existence of such small enough ε is directly from the differentiability definition as $g(x)$ vanishes quickly. Define the map $F : \mathbb{S}^n \times I \rightarrow \mathbb{S}^n$ by

$$F(x, t) = \begin{cases} f(x) & \text{for } 2\varepsilon \leq \|x\|, \\ f(x) - t(2 - \|x\|/\varepsilon)g(x) & \text{for } \varepsilon \leq \|x\| \leq 2\varepsilon, \\ f(x) - tg(x) & \text{for } \|x\| \leq \varepsilon, \end{cases}$$

and let $f_1(x) = F(x, 1)$. Notice, f_1 is the identity near p and as the middle “dimming” function ensures continuity, we have a homotopy $f \simeq f_1$. Also f_1 maps no point other than p to p due to the imposed local condition upon applying Taylor’s Theorem. Thus, there is a small enough open disk about p s.t. if D is the complementary disk in $\mathbb{S}^n$, we have $f_1(D) \subseteq D$ and f_1 still identity on ∂D . Observe, $f_1|_D \simeq \mathbb{1} \text{ rel } \partial D$ with the straight-line homotopy where $D \cong \mathbb{D}^n$. By the Pasting Lemma ([Mu75], p. 108), we can extend this to give a homotopy of f_1 to the identity, therefore $\deg f_1 = \deg \mathbb{1}_{\mathbb{S}^n} = 1$. $\square$

Definition 3.6. The degree in the proposition above is denoted as the **local degree** of f at q .

Definition 3.7. Given two spaces X and Y with basepoints x_0, y_0 , respectively, define their **wedge product** $X \vee Y$ as:

$$X \vee Y = X \sqcup Y / x_0 \sim y_0$$

Proposition 3.15. Let $X = \mathbb{S}_1^n \vee \cdots \vee \mathbb{S}_k^n$, for $n > 0, k > 1$. Then the homomorphism $H_n(\mathbb{S}_1^n) \oplus \cdots \oplus H_n(\mathbb{S}_k^n) \rightarrow H_n(X)$ induced by the inclusions $i_j : \mathbb{S}_j^n \hookrightarrow X$, is an isomorphism. It’s inverse is induced by the projections $X \rightarrow \mathbb{S}_j^n$.

Proof. Immediately follows from the Mayer-Vietoris Theorem. See [Br97], p. 229 for the theorem. $\square$

The following is a crucial yet technical observation of which I have taken some liberties in its presentation.

Proposition 3.16. Let $f : \mathbb{S}^n \rightarrow (Y, y_0)$ be a map. Let $E_1, \dots, E_k$ be disjoint open sets in $\mathbb{S}^n$, each homeomorphic to $\mathbb{R}^n$, $n > 0$. If $f : \left(\mathbb{S}^n - \bigcup_{i=1}^k E_i\right) \mapsto y_0$, then f factors through the space $\mathbb{S}_1^n \vee \cdots \vee \mathbb{S}_k^n$ and it can be “separated” into maps f_j as $f = \sum_{i=1}^k f_j$.

Proof. The proposition requires not a full proof but an elucidation.

Given such a map f , it factors through the space $\mathbb{S}^n / \left(\mathbb{S}^n - \bigcup_{i=1}^k E_i\right) \cong \mathbb{S}_1^n \vee \cdots \vee \mathbb{S}_k^n$

where we have $\mathbb{S}_i^n \cong \mathbb{S}^n / (\mathbb{S}^n - E_i)$, with the maps g and h taking place as follows⁵:

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{f} & Y \\ & \searrow g \quad \nearrow h & \\ & \mathbb{S}_1^n \vee \dots \vee \mathbb{S}_k^n & \end{array}$$

To see the separation, let $i_j : \mathbb{S}_j^n \hookrightarrow \mathbb{S}_1^n \vee \dots \vee \mathbb{S}_k^n$ denote the j th inclusion and $p_j : \mathbb{S}_1^n \vee \dots \vee \mathbb{S}_k^n \rightarrow \mathbb{S}_j^n$ denote the j th projection. Let $g_j = p_j \circ g : \mathbb{S}^n \rightarrow \mathbb{S}_j^n$, and let $h_j = h \circ i_j : \mathbb{S}_j^n \rightarrow Y$, and thus with $f_j = h_j \circ g_j : \mathbb{S}^n \rightarrow Y$ we see that $f = \sum f_j$. $\square$

Theorem 3.17. Given the observation in Proposition 3.16, we have:

$$f_* = \sum_{i=1}^k (f_i)_* : H_n(\mathbb{S}^n) \rightarrow H_n(Y)$$

Proof. With the maps i_j and p_j from the observation, using Proposition 3.15, we can further see that $\sum_j (i_j)_* (p_j)_* = \mathbb{1}_{H_n(\mathbb{S}_1^n \vee \dots \vee \mathbb{S}_k^n)}$. Then for $\alpha \in H_n(\mathbb{S}^n)$, we have $g_*(\alpha) = \sum_j (i_j)_* (p_j)_* g_*(\alpha) = \sum_j (i_j)_* (g_j)_*(\alpha)$. Thus $f_*(\alpha) = h_* g_*(\alpha) = \sum_j h_* (i_j)_* (g_j)_*(\alpha) = \sum_j (h_j)_* (g_j)_*(\alpha) = \sum_j (f_j)_*(\alpha)$. $\square$

Corollary 3.18. Let $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a differentiable map and let $p \in \mathbb{S}^n$ be a regular value with $\{q_1, \dots, q_k\} = f^{-1}(p)$. Let $d_i = \pm 1$ be the local degree of f at q_i , defined and computed as in Proposition 3.14. Let $d = \deg(f)$, then $d = \sum_{i=1}^k d_i$.

Proof. By definition of a regular value, there is a disk D about p such that $f^{-1}(D) = \bigcup_{i=1}^k E_i$, union of disjoint disks about q_i where this union is mapped diffeomorphically to D . Letting r be the antipode of p , we can compose, in the end, f with a deformation of $\mathbb{S}^n$ that stretches D to the whole $\mathbb{S}^n - \{r\}$ so that $\text{int } D$ maps diffeomorphically onto $\mathbb{S}^n - \{r\}$ and $\mathbb{S}^n - \text{int } D$ is mapped onto r (see Figure 4 for my own visualisation). By homotopy invariance, this composition preserves the degree and it falls under the situation described in Proposition 3.16 and Theorem 3.17. $\square$

⁵The interested reader is referred to the visualisation of the diagram in p.192 of [Br97].

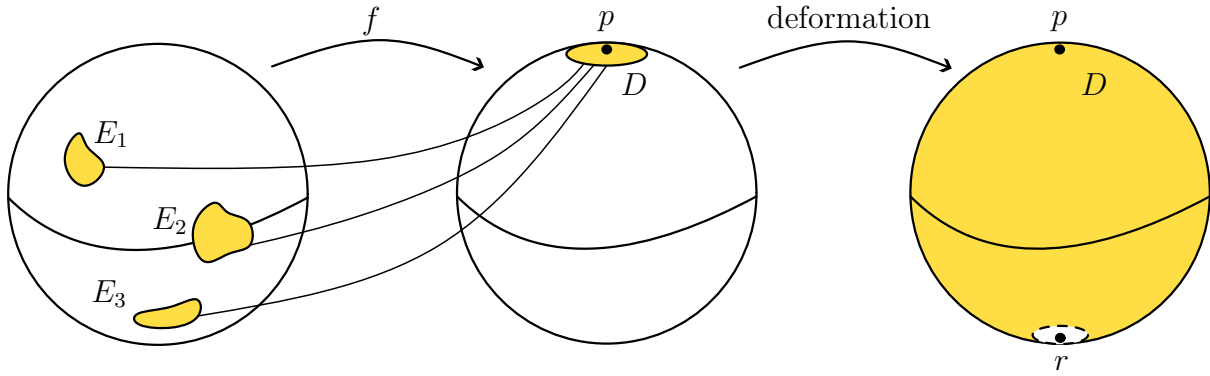


Figure 4: Visualisation on $\mathbb{S}^2$ which clarifies how the observation in Proposition 3.16 applies to the composition

Besides providing us the tool to compute the degree of a map, an important corollary to this theorem is that the degree of a map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is independent of the homology theory used to define it, as long as it is in same coefficients ($\mathbb{Z}$ in our case). However, one might object to this corollary as it begs the question: How, computing the degree of a *differentiable* map, with an *existing regular value*, and a *finite preimage of such regular value*, applies to the continuous map we used to define the degree with homology? Well, using the Whitney Approximation Theorem [Le13], sufficiently, we can find a map smooth around p , homotopic to our continuous map, and by an application of Sard's Theorem [Br97], there exists a regular value in that region. Finiteness of the preimage is actually forced by the Inverse Function Theorem, as the preimage will be discrete, thus with compactness, it will be finite. Therefore, the (homological) degree agrees with the local degree sum on any continuous function $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$.

We shall give at least one example of a degree computation here.

Example 3.1. Consider $\mathbb{S}^1 \subset \mathbb{C}$, and let $\phi_k : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by $\phi_k(z) = z^k$. This map is a differentiable map and all the values are regular. For any point y , the inverse image of is k -many values spaced equally in the domain, with local degree at each as $+1$. Thus $\deg(\phi_k) = k$.

4 CW-Complexes and Cellular Homology

In this section, we give the definition of CW-complexes define cellular homology. We then show its equivalence with singular homology.

4.1 CW-Complexes

Definition 4.1. A **finite CW-complex** K is defined as follows:

1. Let $K^{(0)}$ be a discrete set of points, where its points are called 0-cells,
2. Given $K^{(n-1)}$, let $\{f_{\partial\sigma}\}_{\sigma \in I}$ be a collection of maps $f_{\partial\sigma} : \mathbb{S}^{n-1} \rightarrow K^{(n-1)}$, where I is some indexing set and maps $f_{\partial\sigma}$ are called **attaching maps** for cells σ ,
3. Let Y and B be the disjoint unions, $Y = \sqcup_{\sigma \in I} \mathbb{D}_{\sigma}^n$ and $B = \sqcup_{\sigma \in I} \partial \mathbb{D}_{\sigma}^n = \sqcup_{\sigma \in I} \mathbb{S}_{\sigma}^{n-1}$, where $\mathbb{D}_{\sigma}^n = \mathbb{D}^n$ and $\mathbb{S}_{\sigma}^{n-1} = \mathbb{S}^{n-1}$ for all σ . Put together $f : B \rightarrow K^{(n-1)}$ as $f = \oplus_{\sigma \in I} f_{\partial\sigma}$,
4. Define $K^{(n)} = K^{(n-1)} \cup_f Y$ where $\cup_f$ denotes Y being attached to $K^{(n-1)}$ along f ,
5. Given $K^{(n)}$ for all integers $n \geq 0$, let $K = \bigcup K^{(n)}$ with the **weak topology** where a set $E \subset K$ is open $\iff E \cap K^{(n)}$ is open in $K^{(n)}$ for all n .
6. (Weak Topology). For the topology (of $K^{(n)}$), for each σ let $f_{\sigma} : \mathbb{D}_{\sigma}^n \rightarrow K$ be the canonical maps given by the attaching of the cell σ . Let $K_{\sigma} = \text{im } f_{\sigma}$, a compact subset of K called a **closed cell**. Let $U_{\sigma} = f_{\sigma}(\mathbb{D}_{\sigma}^n - \mathbb{S}_{\sigma}^{n-1})$, denoted an **open cell**. Hence, the topology of each $K^{(n)}$, thus of K , is characterized by: $E \subset K$ open $\iff f_{\sigma}^{-1}(E)$ is open $\iff E \cap K_{\sigma}$ is open in K_{σ} for all σ , where the topology of K_{σ} is the topology of $\mathbb{D}^n / \sim_{f_{\partial\sigma}}$. Maps f_{σ} are called **characteristic maps** of cells σ .
7. (Dimension) If $K^{(m)} = \emptyset$ for some $m \geq 0$, then K is a **finite CW complex** and the maximum $n \geq 0$ s.t. $K^{(n)} \neq \emptyset$ is the **dimension** of the complex.

Remark 4.1. Observe that a closed cell K_{σ} is not generally homeomorphic to $\mathbb{D}^n$ as there are identifications on the boundary. And an open cell U_{σ} is generally not an open subset of K , and it is homeomorphic to interior of $\mathbb{D}^n$. Moreover, in view of the weak topology, a function $g : K \rightarrow X$ for any space X is continuous if and only if each $g \circ f_{\sigma} : \mathbb{D}_{\sigma}^n \rightarrow X$ is continuous.

Definition 4.2. A **subcomplex** A of a CW-complex K is a union of some closed cells of K that is itself a CW-complex with the same attaching maps of cells inherited from K . The subcomplex $K^{(n)}$ of K is called the **n -skeleton** of K .

Proposition 4.1. *CW-complexes are Hausdorff.*

Proof. For a proof, see [Ha02], p. 522. □

We defer the proof for a structural property of CW-complexes to the Appendices, which provides the ‘C’ in the name, as in *closure-finiteness property* (‘W’ denotes the weak topology). This property is used to prove a theorem (A.16) after, which will be implicitly utilized in the degree formula for cellular homology. See Appendix A.3 for these and a refined definition of a CW-complex (Def. A.14).

We give some examples of CW-complexes to later compute their Euler characteristics.

Example 4.1 ($\mathbb{S}^n$). One CW-complex construction on $\mathbb{S}^n$ is by one 0-cell and one n -cell only, with the single attaching map as the unique map $f_{\partial\sigma} : \mathbb{S}^{n-1} \rightarrow \text{point}$.

Another CW-complex on the n -sphere is iterative. Start by two 0-cells, i.e. $K^{(0)} \cong \mathbb{S}^0$. To $K^{(0)}$, attach two 1-cells by the isomorphisms $\mathbb{S}^0 \rightarrow K^{(0)}$, thus $K^{(1)} \cong \mathbb{S}^1$. Continuing in this fashion, we have $K = \bigcup_{i=0}^n K^{(i)} \cong \bigcup_{i=0}^n \mathbb{S}^{(i)}$, the complex with exactly two cells in each dimension.

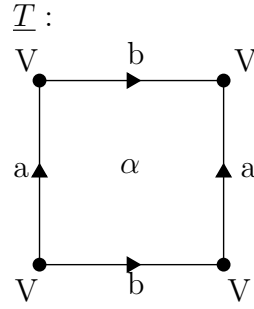
Example 4.2 ($\mathbb{R}P^n$). Consider the iterative CW-complex of $\mathbb{S}^n$, the two cells in each dimension are, homeomorphically:

$$\mathbb{D}_+^i = \{(x_1, \dots, x_{i+1}, 0, \dots, 0) \in \mathbb{S}^n | x_{i+1} \geq 0\}$$

$$\mathbb{D}_-^i = \{(x_1, \dots, x_{i+1}, 0, \dots, 0) \in \mathbb{S}^n | x_{i+1} \leq 0\}$$

Observe that the antipodal map (-1) (introduced in Cor. 3.11) interchanges these two i -cells. Identify these two cells by the map (-1) for each dimension $i = 0, \dots, n$, to get a CW-complex structure on $\mathbb{R}^n P$. The attaching maps deriving from $\mathbb{S}^n$ are identified by the antipodal map, in a similar fashion to the i -cells.

Example 4.3 (T). The CW-complex structure and the identifications on the torus T are all shown in the following diagram:

Figure 5: CW-complex on the Torus T

The attaching maps for the 1-cells are uniquely determined. Considering that the $K^{(1)}$ is a figure-eight (see Figure 3), the attaching map for the one 2-cell is given as $aba^{-1}b^{-1}$.

4.2 n -Cubes and Other Conventions

This subsection is rather a technical adjustment on CW-complexes to transition the characteristic maps on to n -cubes and to determine an orientation of $\mathbb{I}^n$, $\partial\mathbb{I}^n$, and $\mathbb{S}^n$ to have a consistent and working boundary formula in the next subsection. The main reason for n -cubes is that we have $\mathbb{I}^p \times \mathbb{I}^q = \mathbb{I}^{p+q}$, not just homeomorphism like in disks $\mathbb{D}^n$. Despite not using product complexes in this project, this greatly simplifies them and we choose this convention to keep loyal to the main source⁶.

Definition 4.3. Given two pointed spaces $(X, *)$ and $(Y, *)$, define their **smash product** as $X \wedge Y = X \times Y / (X \times \{*\} \cup \{*\} \times Y) = X \times Y / X \vee Y$. Where $\vee$ denotes the wedge product, introduced in Definition 3.7.

Proposition 4.2. If X and Y are compact spaces, then the smash product $X \wedge Y$ is the one-point compactification of the punctured product $(X - \{*\}) \times (Y - \{*\})$. In particular, $\mathbb{S}^p \wedge \mathbb{S}^q \cong \mathbb{S}^{p+q}$.

Proof. For the first statement, we take it for granted from [Br97]. For the particular case, observe that the punctured product of the spheres is $\mathbb{R}^q \times \mathbb{R}^q \cong \mathbb{R}^{p+q}$, where its one-point compactification is $\mathbb{S}^{p+q}$. □

Let $\mathbb{I}^n = \mathbb{I} \times \cdots \times \mathbb{I}$ (n times), with basepoint, and $\mathbb{S}^1 = \mathbb{I}^1 / \partial\mathbb{I}^1$, also a pointed space. Let $\gamma_1 : \mathbb{I}^1 \rightarrow \mathbb{S}^1$ be the quotient map collapsing $\partial\mathbb{I}^1$ to the basepoint of $\mathbb{S}^1$. Let $\mathbb{S}^p = \mathbb{S}^1 \wedge \cdots \wedge \mathbb{S}^1$ (p times), and let $\gamma_p : \mathbb{I}^p \rightarrow \mathbb{S}^p$. Observe $\gamma_{p+q} = \gamma_p \wedge \gamma_q$.

⁶For a treatment of products, refer to pp. 211-214 of [Br97].

Given a homology theory H_* with $\mathbb{Z}$ coefficients, let $\mathbb{I}^0$ be defined as a singleton, so that $H_0(\mathbb{I}^0) \cong \mathbb{Z}$. For this group, choose a generator $[\mathbb{I}^0] \in H_0(\mathbb{I}^0)$. This element is called an **orientation**. Take $\gamma_0 : \mathbb{I}^0 \rightarrow \mathbb{S}^0$ as $0 \mapsto 1$.

We now *orient* $\mathbb{I}^p$, $\mathbb{S}^p$, $\partial\mathbb{I}^p$ inductively. Suppose $\mathbb{I}^p$ is given an orientation by a choice of a generator, then orient $\mathbb{S}^p$ by taking $[\mathbb{S}^p] = \gamma_{p*}[\mathbb{I}^p]$ as the generator, where γ_{p*} is the isomorphism:

$$\gamma_{p*} : H_p(\mathbb{I}^p, \partial\mathbb{I}^p) \rightarrow H_p(\mathbb{S}^p, *)$$

To see where this follows from, see p. 199 of [Br97].

Now, to orient $\partial\mathbb{I}^p$, consider the map $1 \wedge \gamma_p$, that restricts to $\partial\mathbb{I}^{p+1} = (\partial\mathbb{I}^1) \times \mathbb{I}^p \cup \mathbb{I}^1 \times (\partial\mathbb{I}^p) \rightarrow \partial\mathbb{I}^1 \times \mathbb{S}^p \cup \mathbb{I} \times \{*\} \rightarrow \mathbb{S}^0 \wedge \mathbb{S}^p = \mathbb{S}^p$, where the latter equality is the canonical homeomorphism. This map collapses all but the face $\{1\} \times \mathbb{I}^p \subset \mathbb{I}^1 \times \mathbb{I}^p$ to the basepoint, and it is actually γ_p on this face. Now thus by homotopy-equivalence, we have the isomorphism:

$$(1 \wedge \gamma_p)_* : H_p(\partial\mathbb{I}^{p+1}, *) \rightarrow H_p(\mathbb{S}^p, *)$$

and take $[\partial\mathbb{I}^{p+1}] \in H_p(\partial\mathbb{I}^{p+1}, *)$ to go into $[\mathbb{S}^p]$.

We orient $\mathbb{I}^{p+1}$ by choosing the generator it to go into $[\partial\mathbb{I}^{p+1}]$ in the isomorphism $\partial_* : H_{p+1}(\mathbb{I}^{p+1}, \partial\mathbb{I}^{p+1}) \rightarrow H_p(\partial\mathbb{I}^{p+1}, *)$. The order of which we have oriented these homology groups can be roughly visualised in the following diagram of generators, where the squiggly arrow represents the jump to the inductive choice:

$$[\mathbb{I}^0] \xrightarrow{(\gamma_0)_*} [\mathbb{S}^0] \approx \approx \approx [\mathbb{I}^p] \xleftarrow{(\gamma_p)_*} [\mathbb{S}^p] \xleftarrow{(1 \wedge \gamma_p)_*} [\partial\mathbb{I}^{p+1}] \xleftarrow{\partial_*} [\mathbb{I}^{p+1}]$$

We now describe how $\mathbb{I}^n$'s are *inserted* into the complexes through $\mathbb{S}^n$'s. Let K be a complex and for all n -cell σ of K , choose, once and for all, a characteristic map $f_\sigma : \mathbb{I}^n \rightarrow K^{(n)}$ of which we can restrict to $\partial\mathbb{I}^n$ to get the attaching map $f_{\partial\sigma} : \partial\mathbb{I}^n \rightarrow K^{(n-1)}$. Consider $K^{(n)} /_{K^{(n-1)}}$ as pointed space. We have the homeomorphism $K^{(n)} /_{K^{(n-1)}} \cong \vee \mathbb{S}^n$. The projection

$$\mathbb{I}^n \xrightarrow{f_\sigma} K^{(n)} \longrightarrow K^{(n)} /_{K^{(n-1)}} \xrightarrow{\sigma^{\text{th proj}}} \mathbb{S}^n$$

is homeomorphism on the interior and collapses the boundary, so we can assume this to be γ_n . Then denoting the part $K^{(n)} \rightarrow \mathbb{S}^n$ by p_σ , it is the unique map $p_\sigma f_\sigma = \gamma_n : \mathbb{I}^n \rightarrow \mathbb{S}^n$

and for n -cells $\sigma' \neq \sigma$, we have $p_\sigma f_{\sigma'} = \text{constant map to the base point}$.

We use all of these in the end of the next section.

4.3 Cellular Homology and Equivalence with Singular Homology

Using our transition to n -cubes for characteristic maps, we will describe how to compute the homology of a CW-complex. From now on, ∂_* is used instead of ∂_{**} .

Throughout this subsection, let (K, A) be a CW-complex & subcomplex pair. Let $K^{(n)}$ denote $K^{(n)} \cup A$. Let $\oplus_\sigma \mathbb{I}^n$ be the disjoint union of n -disks over the n -cells σ of K not in A . We gather the characteristic maps f_σ (of n -cells over σ not in A) to construct $f : \oplus_\sigma \mathbb{I}^n \rightarrow K^{(n)}$, which restricts to $\oplus_\sigma \partial \mathbb{I}^n \rightarrow K^{(n-1)}$. By additivity of H , have: $H_n(\oplus_\sigma(\mathbb{I}^n, \partial \mathbb{I}^n)) \cong \oplus_\sigma H_n(\mathbb{I}^n, \partial \mathbb{I}^n)$. Thus, this group is a free abelian group on the n -cells of K not in A ⁷. We prove the following to carry this to the n -skeleton of K .

Lemma 4.3. *The induced map $\oplus_\sigma f_{\sigma*} : \oplus_\sigma H_n(\mathbb{I}^n, \partial \mathbb{I}^n) \rightarrow H_n(K^{(n)}, K^{(n-1)})$ is an isomorphism and $H_i(K^{(n)}, K^{(n-1)}) = 0$ for $i \neq n$.*

Proof. By the previous remark with additivity, the lemma is equivalent to the assertion of the isomorphism $f_* : H_*(\oplus_\sigma(\mathbb{I}^n, \partial \mathbb{I}^n)) \rightarrow H_*(K^{(n)}, K^{(n-1)})$. Observe the following diagram:

$$\begin{array}{ccc}
 H_*(\oplus_\sigma(\mathbb{I}^n, \partial \mathbb{I}^n)) & \xrightarrow{f_*} & H_*(K^{(n)}, K^{(n-1)}) \\
 \downarrow \cong & & \downarrow \cong (\text{homotopy-invariance \& 5-lemma}) \\
 H_*(\oplus_\sigma(\mathbb{I}^n, \mathbb{I}^n - \{0\})) & \longrightarrow & H_*(K^{(n)}, K^{(n)} - \bigcup f_\sigma(0)) \\
 \uparrow \cong & & \uparrow \cong (\text{excision \& 5-lemma}) \\
 H_*(\oplus_\sigma(\text{int } \mathbb{I}^n, \text{int } \mathbb{I}^n - \{0\})) & \xrightarrow{\cong} & H_*(K^{(n)} - K^{(n-1)}, K^{(n)} - K^{(n-1)} - \bigcup f_\sigma(0))
 \end{array}$$

First, the lower and upper squares commute because the maps commute pre-homology (hence by functoriality, it follows); all the vertical maps are inclusions and horizontal maps are f and its restrictions. Thus the diagram commutes.

Next, for the upper vertical isomorphisms (also for the left column), the following (ro-

⁷See (D_n) of Theorem 3.5.

tated) ladder of homology exact sequences illustrates the reason:

$$\begin{array}{ccc}
 \begin{array}{c} \vdots \\ \partial_* \downarrow \\ H_i(K^{(n)} - \bigcup f_\sigma(0)) \\ \downarrow \\ H_i(K^{(n)}) \\ \downarrow \\ H_i(K^{(n)}, K^{(n)} - \bigcup f_\sigma(0)) \\ \partial_* \downarrow \\ H_{i-1}(K^{(n)} - \bigcup f_\sigma(0)) \\ \downarrow \\ H_{i-1}(K^{(n)}) \\ \downarrow \\ \vdots \end{array} & \begin{array}{c} \xleftarrow{\cong(\text{homotopy-inv.})} \\ \xleftarrow{\cong} \\ \xleftarrow{\cong(5\text{-lemma})} \\ \xleftarrow{\cong(\text{homotopy-inv.})} \\ \xleftarrow{\cong} \end{array} & \begin{array}{c} \begin{array}{c} \vdots \\ \partial_* \downarrow \\ H_i(K^{(n-1)}) \\ \downarrow \\ H_i(K^{(n)}) \\ \downarrow \\ H_i(K^{(n)}, K^{(n-1)}) \\ \partial_* \downarrow \\ H_{i-1}(K^{(n-1)}) \\ \downarrow \\ H_{i-1}(K^{(n)}) \\ \downarrow \\ \vdots \end{array} \end{array}
 \end{array}$$

The lower-vertical isomorphisms due to excision are straightforward. The *bridging* bottom horizontal isomorphism is due to the map of spaces being a homeomorphism by definition of a CW-complex. Therefore, f_* is also an isomorphism. $\square$

Commutativity of the following diagram is implicitly used in cellular boundary formula derivation, which states that it does not matter whether you first glue an n -disk (n -cube) and take the boundary, or first take the boundary and glue.

Corollary 4.4 (of Proposition 3.8). *The following diagram commutes:*

$$\begin{array}{ccc}
 H_n(K^{(n)}, K^{(n-1)}) & \xrightarrow{\partial_*} & \tilde{H}_{n-1}(K^{(n-1)}, A) \\
 \oplus_\sigma f_{\sigma*} \downarrow & & \downarrow \oplus_\sigma f \partial_{\sigma*} \\
 \oplus_\sigma H_n(\mathbb{I}^n, \partial \mathbb{I}^n) & \xrightarrow{\oplus_\sigma \partial_*} & \oplus_\sigma \tilde{H}_{n-1}(\partial \mathbb{I}^n)
 \end{array}$$

Proof. It follows by the naturality of the connecting homomorphism of a triple in Proposition 3.8. $\square$

We observe that by Lemma 4.3 and by the exact homology sequence of $(K^{(n)}, K^{(n-1)})$, we have $H_i(K^{(n-1)}, A) \rightarrow H_i(K^{(n)}, A)$ as onto for $i \neq n$, as one-to-one for $i \neq n-1$. Hence, it is an isomorphism for $i \neq n$ or $i \neq n-1$. By induction on n (keeping i fixed), we have:

$$H_i(K^{(n)}, A) = 0 \quad \text{for } n < i \text{ (and for } i < 0).$$

Then this brings us the exact sequence:

$$0 \longrightarrow H_n(K^{(n)}, A) \longrightarrow H_n(K^{(n)}, K^{(n-1)}) \xrightarrow{\partial_*} H_{n-1}(K^{(n-1)}, A) \longrightarrow H_{n-1}(K^{(n)}, A) \longrightarrow 0$$

Observe the following diagram, on which we define β_{n+1} by commutativity:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ H_{n+1}(K^{(n+1)}, A) & \xrightarrow{j_{n+1}} & H_{n+1}(K^{(n+1)}, K^{(n)}) & \xrightarrow{\partial_{n+1}} & H_n(K^{(n)}, A) & \xrightarrow{i_n} & H_n(K^{(n+1)}, A) \longrightarrow 0 \\ & & \searrow \beta_{n+1} & & \downarrow j_n & & \\ & & & & H_n(K^{(n)}, K^{(n-1)}) & & \\ & & & & \downarrow \partial_n & & \\ & & & & \tilde{H}_{n-1}(K^{(n-1)}, A) & & \end{array}$$

It follows, $\beta_{n+1} = j_n \circ \partial_{n+1}$, and thus by $\beta_n \beta_{n+1} = j_{n-1} \partial_n j_n \partial_{n+1} = j_{n-1} \circ 0 \circ \partial_{n+1} = 0$, we have a boundary operator β . From $\ker \beta_{n+1} = \ker \partial_{n+1} = \text{im } j_{n+1}$, we also have $\ker \beta_n = \ker \partial_n = \text{im } j_n$ and $\text{im } \beta_{n+1} = j_n(\text{im } \partial_{n+1})$, with j_n as a monomorphism on $H_n(K^{(n)})$. As a result, j_n induces the following:

$$\begin{array}{ccc} H_n(K^{(n)}, A) & \xrightarrow{\cong} & \ker \beta_n \\ \cup & & \cup \\ \text{im } \partial_{n+1} & \xrightarrow{\cong} & \text{im } \beta_{n+1} \end{array}$$

and also

$$H_n(K^{(n+1)}, A) \cong \text{coker } \partial_{n+1} \xrightarrow{\cong} \ker \beta_n / \text{im } \beta_{n+1}$$

But since $H_n(K^{(i)}, K^{(i-1)}) = 0$ for $i > n$, and since the exact sequences, we have:

$$H_n(K^{(n+1)}, A) \xrightarrow{\cong} H_n(K^{(n+2)}, A) \xrightarrow{\cong} H_n(K^{(n+3)}, A) \xrightarrow{\cong} \dots$$

Thus, if we have $\dim(K) < \infty$, we get the following isomorphism:

$$H_n(K, A) \cong H_n(K^{(n+1)}, A) \xrightarrow{\cong} \ker \beta_n / \text{im } \beta_{n+1}$$

Thus, we are able to compute the relative homology of the complex pair (K, A) , $H_n(K, A)$, from the chain complex with the n th term as $H_n(K^{(n)}, K^{(n-1)})$, and boundary operator as β . We now will finalize discussion by a closer look on this chain complex, which is

denoted as the *cellular chain complex*.

Denote the free abelian group based on the n -cells of K not in A , with $C_n(K, A)$. Define the homomorphisms:

$$C_n(K, A) \begin{array}{c} \xrightarrow{\Psi} \\ \xleftarrow{\Phi} \end{array} H_n(K^{(n)}, K^{(n-1)})$$

given by:

$$\Psi \left(\sum_{\sigma} n_{\sigma} \sigma \right) = \sum_{\sigma} n_{\sigma} f_{\sigma*} [\mathbb{I}^n] \text{ and } \Phi(\alpha) = \sum_{\sigma} \phi(p_{\sigma*}(\alpha)) \sigma$$

Here $\phi_n : H_n(\mathbb{S}^n, \rightarrow \mathbb{Z})$ is the homomorphism given uniquely by $\phi_n[\mathbb{S}^n] = 1$. Ψ is already shown to be an isomorphism in Lemma 4.3. We'd like to show that $\Phi \circ \Psi = 1$, i.e. Φ is the inverse of Ψ . Given a basis element n -cell σ of $C_n(K, A)$, the composition yields: $\Phi(\Psi(\sigma)) = \Phi(f_{\sigma*}[\mathbb{I}^n])\sigma = \phi_n([\mathbb{S}^n])\sigma = 1\sigma = \sigma$. Through these explicit maps we compute the homology as follows. Define the boundary operator ∂ on $C_*(K, A)$ by the commutativity of the diagram:

$$\begin{array}{ccccc} C_{n+1}(K, A) & \xrightarrow{\quad \partial_{n+1} \quad} & C_n(K, A) & & \\ \downarrow \cong & & \downarrow \cong & & \\ H_{n+1}(K^{(n+1)}, K^{(n)}) & \xrightarrow{\partial_{n+1}} H_n(K^{(n)}) & \xrightarrow{j_n} & H_n(K^{(n)}, K^{(n-1)}) & \\ & \searrow \beta_{n+1} \nearrow & & & \end{array}$$

Here the vertical isomorphisms are those given above. Given a generator $\sigma \in C_{n+1}(K, A)$, trace the diagram as follows where $[\tau : \sigma] \in \mathbb{Z}$ are integer coefficients that make the diagram commute:

$$\begin{array}{ccc} \sigma & \xrightarrow{\quad \quad \quad} & \sum_{\tau} [\tau : \sigma] \tau \\ \updownarrow & & \updownarrow \\ (f_{\sigma})_*[\mathbb{I}^{n+1}] & \xrightarrow{\partial_*} (f_{\partial\sigma})_*[\partial\mathbb{I}^{n+1}] & \xrightarrow{\quad \quad \quad} (f_{\partial\sigma})_*[\partial\mathbb{I}^{n+1}] \end{array}$$

By our previous description of the vertical map on the right, we can calculate the **inci-**

dence number $[\tau : \sigma]$ as:

$$\begin{aligned}
 [\tau : \sigma] &= \phi_n(p_{\tau*}(f_{\partial\sigma*}[\partial\mathbb{I}^{n+1}])) \\
 &= \phi_n((p_{\tau}f_{\partial\sigma})_*[\partial\mathbb{I}^{n+1}]) \\
 &= \phi_n(\deg(p_{\tau}f_{\partial\sigma})[\mathbb{S}^n]) \\
 &= \deg(p_{\tau}f_{\partial\sigma})
 \end{aligned}$$

by using the definition of the degree and the choice of the *orientations* in Subsection 4.2.

Ultimately we have proven the following.

Theorem 4.5. $H_*(K, A)$ is isomorphic to the homology of the chain complex $C_*(K, A)$ where the boundary operator $\partial : C_{n+1}(K, A) \rightarrow C_n(K, A)$ is given by $\partial\sigma = \sum_{\tau} [\tau : \sigma]\tau$ where

$$[\tau : \sigma] = \deg(p_{\tau}f_{\partial\sigma}).$$

Finally, we have established isomorphism with what we defined as cellular homology and singular homology. We will, for the conciseness of this draft, restrict ourselves to one example.

Example 4.4. Consider the CW-complex structure on $\mathbb{R}P^n$, particular to $n = 2$, given in the Example 4.2. The way to attach the 1-cell onto the 0-cell is unique; making $K^{(1)}$ into a circle. Attaching of the 2-cell to this is done by identifying antipodal points on $\mathbb{S}^1$. The chain complex $C_*(K)$ is:

$$\dots \longrightarrow 0 \longrightarrow C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K) \longrightarrow 0 \longrightarrow \dots$$

which is homeomorphically:

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

For the 1-cell, we have $\partial a = x - x = 0$, where x is the only 0-cell. For the 2-cell, σ , we can think of the attaching map as the complex attaching map $z \mapsto z^2$. Recalling again the convention subsection (4.2), this is $p_{\sigma}f_{\partial\sigma}$ up to a difference of orientation (which just changes the sign $[a : \sigma]$). Thus, as in Example 3.1, this has degree 2. Here, the neglect in

orientation won't change the groups as the chain complex looks like:

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

Thus $H_2(\mathbb{R}P^2) = 0$, $H_1(\mathbb{R}P^1) \cong \mathbb{Z}_2$, and $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$.

Remark 4.2. Establishing isomorphism between cellular and singular homology groups - when coincides - extends to simplicial homology too as it is just a particular case. Thus for homology group calculations, if it exists, we can use the simplicial homology computations just like in Example A.6.

5 Proof

In this section, we finally equate the combinatorial definition of the Euler characteristic to a homological definition. To accomplish this, we need to introduce some algebra.

Theorem 5.1 (The fundamental theorem of finitely generated abelian groups, [DuFo04], pp. 158-159). Let G be a finitely generated abelian group. Then

$$(1) \quad G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_s},$$

for some integers $r, n_1, n_2, \dots, n_s$ satisfying the following conditions:

- (a) $r \geq 0$ and $n_j \geq 2$ for all j , and
 - (b) $n_i | n_{i+1}$ for $1 \leq i \leq s-1$
- (2) the expression in (1) is unique, that is, if $G \cong \mathbb{Z}^t \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_u}$ satisfy (a) and (b), then $t = r$, $u = s$, and $m_i = n_i$ for all i .

Proof. See Appendix A.1. □

The following parts are from [Br97], p. 215.

Definition 5.1. The integer r in the fundamental theorem is called the **(free) rank** or the **Betti number** of the finitely generated abelian group G .

Proposition 5.2. Let A, B , and C be finitely generated abelian groups and

$$0 \longrightarrow B \xrightarrow{i} A \xrightarrow{j} C \longrightarrow 0$$

be a short exact sequence. Then $\text{rank}(A) = \text{rank}(B) + \text{rank}(C)$.

Proof. Follows from Proposition A.7 applied to G as a $\mathbb{Z}$ -module and by the fundamental theorem (5.1). $\square$

Definition 5.2. Given a space X , if $H_i(X)$ is finitely generated for each i , the space X is said to be of **finite type**. Further if $H_i(X) = 0$ for all but finitely many i , X is said to be of **bounded finite type**.

Definition 5.3. Given a space X of bounded finite type, redefine its **Euler characteristic** $\chi(X)$ as:

$$\chi(X) = \sum_i (-1)^i \text{rank } H_i(X)$$

Here is the main theorem this project set out to prove. We recall the combinatorial definition given in Definition 1.1.

Theorem 5.3 (Euler-Poincaré). Let X be a finite CW-complex and let a_i be the number of i -cells in X . Take the definition of $\chi(X)$ as in Definition 5.3, then:

$$\chi(X) = \sum_i (-1)^i a_i$$

Proof. We know, with the theory so far, that $a_i = \text{rank } C_i(X)$. Denote the group of i -cycles with $Z_i \subset C_i = C_i(X)$, group of i -boundaries $B_i = \partial C_i$, and the homology $H_i = H_i(C_*(X)) \cong H_i(X)$. Thus obtaining $H_i = Z_i/B_i$. From Proposition 5.2, the exact sequence:

$$0 \longrightarrow Z_i \longrightarrow C_i \xrightarrow{\partial} B_{i-1} \longrightarrow 0$$

gives $\text{rank}(C_i) = \text{rank}(Z_i) + \text{rank}(B_{i-1})$. We also have $\text{rank}(Z_i) = \text{rank}(B_i) + \text{rank}(H_i)$ from the exact sequence $0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0$. We can add these equations with signs $(-1)^i$ to get:

$$\begin{aligned} \sum_i (-1)^i (\text{rank}(H_i) + \text{rank}(B_i)) &= \sum_i (-1)^i \text{rank}(Z_i) \\ &= (-1)^i (\text{rank}(C_i) - \text{rank}(B_{i-1})). \end{aligned}$$

With the terms in B_* cancelling, we get $\chi(X) = \sum_i (-1)^i \text{rank}(H_i) = \sum_i (-1)^i \text{rank}(C_i) =$

$$\sum_i (-1)^i a_i.$$

□

We can now prove Euler's Polyhedron Formula from the introduction, as all convex polyhedra is homeomorphic to the 2-sphere, by means of projection.

Corollary 5.4. *For any CW-complex structure on the 2-sphere with F 2-cells, E 1-cells, V 0-cells, we have $F - E + V = 2$.*

We can now look at some examples. Immediatly we can see from the simplicial homology calculations of Example A.6, we have the characteristics:

$$\begin{array}{ll} \chi(T) = 0 & \chi(\mathbb{R}P^2) = 1 \\ \chi(K) = 0 & \chi(\mathbb{S}^1) = 0 \end{array}$$

Additionally, from the cellular homology computation of its homology groups, we have the Euler characteristic $\chi(\mathbb{R}P^n)$ for the real projective space.

$$\chi(\mathbb{R}P^n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

We have now proven that if any of these spaces are sufficiently homotopically deformed into another, their Euler characteristics will be left the same.

A Appendices

A.1 Modules over Principal Ideal Domains

Remark A.1. This subsection is devoted to proving the The Fundamental Theorem of Finitely Generated Abelian Groups (Theorem 5.1) as a corollary of the structure theorem for modules over principal ideal domains (P.I.D.s) [DuFo04]. For the relevance of the theory to the project, this direction is chosen over the proof where the Smith normal form is explicitly utilized.

Definition A.1. An R -module F is said to be **free** on the subset A , if for all nonzero $x \in F$, there exist unique $r_1, \dots, r_n \in R$ nonzero, and unique $a_1, \dots, a_n \in A$ such that $x = r_1 a_1 + \dots + r_n a_n$, for some $n \in \mathbb{Z}^+$. A is called a **basis** for F . If R is a commutative ring, $|A|$ is called the **rank** of F .

Definition A.2. The free $\mathbb{Z}$ -module on a set A is called the **free abelian group on A** and it is given by:

$$\mathbb{F}(A) \cong \bigoplus_{a \in A} \mathbb{Z}$$

If $|A| = n$, $\mathbb{F}(A)$ is called the free abelian group of rank n .

Definition A.3. The left R -module M is called a **Noetherian R -module** if it satisfies the ascending chain condition on submodules, i.e. whenever

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$$

is an ascending chain of submodules of M , $\exists m \in \mathbb{Z}^+$ such that $\forall k \geq m$, $M_k = M_m$.

A ring R is said to be Noetherian if it is a Noetherian R -module.

Theorem A.1. Let R be a ring and let M be a left R -module. Then TFAE:

- (1) M is a Noetherian R -module.
- (2) Every nonempty set of submodules of M contains a maximal element under inclusion.
- (3) Every submodule of M is finitely generated.

Proof. [(1) implies (2)] Assume M is Noetherian. We shall show the contrapositive. Let Σ be a nonempty set of submodules of M , choose any $M_1 \in \Sigma$. If it is maximal, we are done. If not, then there exists M_2 such that $M_1 \subset M_2$ properly. Proceeding, by invoking the Axiom of Choice, (Dependent Choice is enough to be more accurate), one can see that there has to be a maximal element.

[(2) implies (3)] Assume (2). Let N be a submodule of M . Let Σ be the set of all finitely generated submodules of N . Since $\{0\} \in \Sigma$, Σ is nonempty. Thus, it contains a maximal element, call it $K \in \Sigma$. If $N = K$, we are done. Suppose not, and let $x \in N - K$. Since

K is also finitely generated, $\langle K, [x] \rangle$ is also finitely generated and it is a finitely generated submodule of N , contradicting the maximality of K . Therefore, we must have $K = N$.

[(3) implies (1)] Assume (3). Let $M_1 \subseteq M_2 \subseteq M_3 \dots$ be an ascending chain of submodules of M . Let

$$T := \bigcup_{i=1}^{\infty} M_i$$

T is a submodule itself. By (3), T is finitely generated. Let the generating set be $\{a_1, a_2, \dots, a_n\}$. By definition of T , we have, each a_i lying in one of the submodules in the chain M_{j_i} . Let $m = \max \{j_1, \dots, j_n\}$. Then $a_i \in M_m$, so that $N \subseteq M_m$. This implies that for all $k \geq m$, $M_m = N = M_k$. $\square$

Corollary A.2. *If R is a P.I.D., then every nonempty set of ideals of R has a maximal element and R is a Noetherian ring.*

Proof. The P.I.D. R satisfies condition (3) above with $M = R$. $\square$

Remark A.2. As to elucidate the necessity for P.I.D.s, it is true that a module over a Noetherian ring is Noetherian iff it is finitely generated.⁸ Then in the general setting of arbitrary rings, the condition that M being a Noetherian R -module is stronger than the condition that it is finitely generated.

Proposition A.3. *Let R be an integral domain and let M be a free R -module of rank $n < \infty$. Then any $n + 1$ elements of M are R -linearly dependent.*

Proof. Embed R in its quotient field F (R is an integral domain). Since $M \cong R \oplus R \oplus \dots \oplus R$, we have $M \subseteq F \oplus F \oplus \dots \oplus F$, where the latter is an n -dimensional vector space, so any $n + 1$ elements of M are F -linearly dependent. To obtain the R -linear dependence, we can multiply through by the product of all the denominators. $\square$

Definition A.4. For any integral domain R the **rank** of an R -module M is the maximum number of R -linearly independent elements of M .

Remark A.3. The preceding proposition shows that the definition of rank of a module agrees with the previous definition of rank and it is also consistent with the linear-algebraic notion of basis when $R = F$ is a field.

⁸The proof for this is out of the scope of this project at the present.

Definition A.5. If R is any integral domain and M is any R -module

$$\text{Tor}(M) := \{x \in M \mid rx = 0 \text{ for some nonzero } r \in R\}$$

is a submodule of M . It is called **the torsion submodule of M** . If N is any submodule of $\text{Tor}(M)$, N is called a torsion submodule of M . If $\text{Tor}(M) = 0$, M is called **torsion-free**.

An R -module is called a **torsion module** if $M = \text{Tor}(M)$.

Definition A.6. For any submodule N of M , **the annihilator** of N is the ideal of R defined by

$$\text{Ann}(N) := \{r \in R \mid rn = 0 \text{ for all } n \in N\}.$$

There are some claims that are stated with only the ideas of proofs for the sake of brevity and as their proofs are more of a mechanical nature. The claims are used throughout the main theorems in the appendix.

Proposition A.4. *Let M be a module over an integral domain R . Given a nonzero torsion element x in M , $\{x, 0\}$ is a linearly dependent set. Therefore, $\text{Tor}(M)$ has rank 0. Moreover, $\text{rank}(M) = \text{rank}\left(M/\text{Tor}(M)\right)$.*

Proof. Follows from the definitions. □

Proposition A.5. *Let M be a module over an integral domain R of rank n . Let $\{x_1, \dots, x_n\}$ be a maximal linearly independent set in M . Let $N = Rx_1 + \dots + Rx_n$ be the submodule generated by the set $\{x_1, \dots, x_n\}$. Then $N \cong R^n$ and M/N is a torsion R -module.*

Conversely, if M contains a submodule N free of rank n and M/N is a torsion R -module, then M has rank n .

Proof. The isomorphism $N \cong R^n$ follows from the canonical R -linear isomorphism. M/N being a torsion R -module follows from finding an annihilating element $r \in R$ for $m \in M$ such that $r(m + N) = 0$, using linear dependence.

Converse direction follows by a bounding argument. □

Proposition A.6. *Let A and B modules of ranks m and n , respectively, over an integral domain R . Then $\text{rank}(A \oplus B) = m + n$.*

Proof. Follows from Proposition A.5. □

Proposition A.7. *Let M be a module over an integral domain R of rank m . Let N be a submodule of N rank n . Suppose that M/N has rank s . Then $m = n + s$.*

Proof. The proof follows by explicitly configuring maximal linearly independent elements. □

Proposition A.8. *Let $A_i, i = 1, \dots, n$ be modules over a ring R . Let B_i be a submodule of A_i . Then:*

$$(A_1 \oplus \dots \oplus A_n) / (B_1 \oplus \dots \oplus B_n) \cong A_1/B_1 \oplus \dots \oplus A_n/B_n$$

Proof. Follows from the canonical surjection and the First Isomorphism Theorem of modules. □

Theorem A.9. Let R be a P.I.D., let M be a free R -module of finite rank α and let N be a submodule of M . Then

- (1) N is free of rank $\beta \leq \alpha$ and
- (2) there exists a basis $y_1, \dots, y_\alpha$ of M so that $a_1 y_1, \dots, a_\beta y_\beta$ is a basis of N where $a_1, \dots, a_\beta$ are nonzero elements of R with the divisibility relations

$$a_1 | a_2 | \dots | a_m.$$

Proof. Let R be a P.I.D., M be a free R -module of finite rank of α , and N be a submodule of M .

Firstly, observe that if $N = \{0\}$, then the theorem holds trivially. Suppose $N \neq \{0\}$.

For each $\varphi \in \text{Hom}_R(M, R)$, observe that $\varphi(N)$ is a submodule of R , in particular it is an ideal of R . Since R is a P.I.D. denote $\varphi(N) := (a_\varphi)$ and define $\Sigma := \{(a_\varphi) \mid \varphi \in \text{Hom}_R(M, R)\}$.

Since $(0) \in \Sigma$, Σ is nonempty, hence by Corollary A.2, it has a maximal element; denote it $\nu \in \Sigma$ and let $\nu(N) := (a_1)$. $y \in N$ be the element s.t. $\nu(y) = a_1$.

Want to show that $a_1 \neq 0$. Let $x_1, \dots, x_\alpha$ be any basis of M . Let $\pi_i \in \text{Hom}_R(M, R)$ be the projection homomorphism onto the i th component w.r.t. to the given basis. Then as $N \neq \{0\}$, there exists i such that $\pi_i(N) \neq 0$, and as (a_1) is maximal in Σ , it is nonzero, hence $a_1 \neq 0$.

Then want to show that $a_1|\varphi(y)$ for all $\varphi \in \text{Hom}_R(M, R)$. Let $(d) := \langle a_1, \varphi(y) \rangle$. Then $d|a_1$ and $d|\varphi(y)$ in R . Let $d = r_1a_1 + r_2\varphi(y)$ for some $r_1, r_2 \in R$.

Consider $\psi := r_1\nu + r_2\varphi$. Observe that $\psi(y) = (r_1\nu + r_2\varphi)(y) = r_1\nu(y) + r_2\varphi(y) = r_1a_1 + r_2\varphi(y) = d$. This means $d \in \psi(N)$, which in turn means $(d) \subseteq \psi(N)$. Since also $d|a_1$, we have $(a_1) \subseteq (d)$, and by the maximality of (a_1) in Σ , we have equality $(a_1) = (d) = \psi(N)$. In particular, since d divides $\varphi(y)$, from $(a_1) = (d)$, it is seen that $a_1|\varphi(y)$.

Applying the result above, we have $a_1|\pi_1(y)$ for all i . Write $\pi_1(y) = a_1b_i$ for some b_i and define $y_1 = \sum_{i=1}^{\alpha} b_ix_i$. Observe that $a_1y_1 = y$. Since $a_1 = \nu(y) = \nu(y_1a_1) = a_1\nu(y_1)$ and $a_1 \neq 0$, by R being an integral domain, we obtain $\nu(y_1) = 1$. Now we want to show:

- (a) $M = Ry_1 \oplus \ker \nu$, and
- (b) $N = Ry_1a_1 \oplus (N \cap \ker \nu)$.

To see (a), take $x \in M$ and write: $x = \nu(x)y_1 + (x - \nu(x)y_1)$. Observe:

$$\nu(x - \nu(x)y_1) = \nu(x) - \nu(x)\nu(y_1) = 0.$$

Therefore, we can decompose into a sum. To see that the sum is direct, take $ry_1 \in Ry_1 \cap \ker \nu$. By definition, $\nu(ry_1) = \nu(r)\nu(y_1) = \nu(r) = 0$. Thus the intersection is empty.

To see (b), by definition of a_1 being the generator of the maximal ideal, observe that $a_1|\nu(x')$ for all $x' \in N$. Letting writing $\nu(x') = a_1b$ for some $b \in R$, we see, by the same logic utilized in (a), that:

$$x' = \nu(x')y_1 + (x' - \nu(x')y_1) = ba_1y_1 + (x' - ba_1y_1)$$

This establishes the sum; the directness is a particular case of the directness in (a).

Now (1) is proved using induction on the rank of N , β . If $\beta = 0$, then N is a torsion module. But since a free module is torsion free, (1) holds trivially. Suppose $\text{rank}(N) = \beta > 0$ and all submodules of rank $< \beta$ are free. Then by Proposition A.6, since the sum in (b) is direct, $N \cap \ker \nu$ has rank $\beta - 1$. By the induction hypothesis, it is then free of rank $\beta - 1$. By directness again, given any basis of $N \cap \ker \nu$, we can adjoin a_1y_1 to it to obtain a basis for N . Hence N is free of rank β .

(2) is proved using induction on the rank of M , α . If $\alpha = 1$, then a basis for the submodule can be found and the divisibility relations are satisfied trivially. Suppose $\alpha > 0$ and suppose (2) holds for all $N \leq M$ submodules, such that $\text{rank}(N) < \text{rank}(M)$. Applying (1) to $\ker \nu$, we see that it is free and as the sum in (a) is direct, it is of rank $\alpha - 1$. Applying the induction hypothesis to $\ker \nu$ and $N \cap \ker \nu$, we have, given a basis $y_2, \dots, y_\alpha$ for $\ker \nu$, there exists $a_2, \dots, a_\beta \in R$ nonzero such that $a_2 y_2, \dots, a_\beta y_\beta$ is a basis for $N \cap \ker \nu$, with $a_2 | \dots | a_\beta$. Since the sums in (a) and (b) direct, $y_1, \dots, y_\alpha$ and $a_1 y_1, \dots, a_\beta y_\beta$ are bases of M and N , respectively. To complete the induction, it suffices to show that $a_1 | a_2$. In ideal-theoretic terms, this amounts to showing that $a_2 \in (a_1)$. Let $\varphi \in \text{Hom}_R(M, R)$ be such that $\varphi(y_1) = \varphi(y_2) = 1$. Then $\varphi(a_1 y_1) = a_1 \Rightarrow a_1 \in \varphi(N) \Rightarrow (a_1) \subseteq \varphi(N)$. By maximality of (a_1) in Σ , $(a_1) = \varphi(N)$. Since also $a_2 \in \varphi(N)$, $a_2 \in (a_1)$, so that $a_1 | a_2$. $\square$

Definition A.7. Let R be any ring. Then an R -module C is called **cyclic** if $\exists x \in C$ such that $C = Rx$.

Remark A.4. Given a cyclic R -module C over a ring R , we can write the surjection $\pi : R \rightarrow C$ by $r \mapsto rx$. By the First Isomorphism Theorem for modules, we have: $R / \ker \pi \cong C$. If R is a P.I.D., then C is of the form $R / (a)$ where $(a) = \text{Ann}(C)$.

Theorem A.10 (Structure Theorem: Existence of the Invariant Factor Form). Let M be a finitely-generated module over a P.I.D. R . Then:

- (1) M is isomorphic to finitely many direct sum of cyclic modules, i.e.

$$M \cong R/(a_1) \oplus \cdots \oplus R/(a_m) \oplus R^r$$

where $r \in \mathbb{N}$ and $a_1, \dots, a_m \in R$ nonzero and non-unit in R , with the relation:

$$a_1 | a_2 | \cdots | a_m$$

- (2) M is torsion free if and only if M is free,

- (3) In the decomposition in (1), we have:

$$\text{Tor}(M) \cong R/(a_1) \oplus \cdots \oplus R/(a_m)$$

Proof. Suppose the premises in the theorem. Let $\{x_1, \dots, x_n\}$ be a minimal finite set that generates M . Let R^n be the free R -module of rank n with a basis $\{b_1, \dots, b_n\}$. By the surjection $\pi : R^n \rightarrow M$ given by $\pi(b_i) = x_i$, and the First Isomorphism Theorem for modules, we have $R^n / \ker \pi \cong M$. By Theorem A.9 applied to R^n and $\ker \pi$, we have a basis $y_1, \dots, y_n$ of R^n such that $a_1 y_1, \dots, a_m y_m$ is a basis for $\ker \pi$ for some $m \leq n$, and nonzero $a_1, \dots, a_m \in R$ with the consecutive divisibility condition. Thus we have:

$$M \cong R^n / \ker \pi \cong (Ra_1 \oplus \cdots \oplus Ra_m) / (Ra_1 y_1 \oplus \cdots \oplus Ra_m y_m)$$

To be able to identify the quotient on the right side, consider the natural R -module surjection φ :

$$\begin{aligned} \varphi : Ry_1 \oplus \cdots \oplus Ry_n &\longrightarrow R/(a_1) \oplus \cdots \oplus R/(a_m) \oplus R^{n-m} \\ (k_1 y_1, \dots, k_n y_n) &\mapsto (k_1 \bmod (a_1), \dots, k_m \bmod (a_m), k_{m+1}, \dots, k_n) \end{aligned}$$

By Proposition A.8 and the First Isomorphism Theorem for modules, we have:

$$M \cong R/(a_1) \oplus \cdots \oplus R/(a_m) \oplus R^{n-m}$$

Removing any zero $R/(a_i)$ due to a_i being a unit, we arrive at the decomposition in (1) with $r = n - m$.

$R/(a)$ is a torsion R -module for any nonzero $a \in R$, therefore M is torsion free if and only if $M \cong R^r$, i.e. free, thus (2).

(3) follows directly from the results above and the aforementioned definitions. $\square$

Definition A.8. The integer r in the Structure Theorem is called the **free rank** or the **Betti number** of the module M . The elements $a_1, \dots, a_m$ of R are called the **invariant factors** of M (defined up to multiplication by units).

The decomposition in the Structure Theorem can be further refined to have simpler annihilators of cyclic modules, i.e. (0) or (p^k) where p a prime, using the Chinese Remainder Theorem.

Theorem A.11 (Structure Theorem: Existence of the Elementary Divisor Form). Let M be a finitely-generated module over a P.I.D. R . Then M is direct sum of finitely many cyclic modules whose annihilators are either (0) or generated by powers of primes, i.e.

$$M \cong R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_m^{\alpha_m}) \oplus R^r$$

where $r \in \mathbb{N}$ and $p_1^{\alpha_1}, \dots, p_m^{\alpha_m}$ are positive powers of, not necessarily distinct, primes in R .

Proof. Let a be a nonzero element in the P.I.D. R . In particular, since R is also a Unique Factorization Domain (U.F.D.), for some unit $u \in R$ and distinct primes $p_i \in R$, $i = 1, \dots, n$, we can write:

$$a = up_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

Since this factorization is unique up to units, the ideals $p_i^{\alpha_i}$ are also defined uniquely. Observe that for any $i \neq j$, $(p_i^{\alpha_i}) + (p_j^{\alpha_j}) = R$, as $\gcd(p_i, p_j) = 1$. In other words, the ideals $(p_i^{\alpha_i})$ are comaximal in pairs. The intersection over all these ideals is the ideal (a) as it is the least common multiple of these prime powers. Therefore, by the Chinese Remainder Theorem, we have:

$$R/(a) \cong R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_m^{\alpha_m})$$

Applying this results to the modules in the Invariant Factor Form proves the theorem. $\square$

Definition A.9. Let M be a module over a P.I.D. R , then the prime powers $p_1^{\alpha_1}, \dots, p_m^{\alpha_m}$ (defined up to multiplication by units) are called the **elementary divisors** of the module M .

Lemma A.12. Let R be a P.I.D. and let p be a prime in R . Denote the field $F := R/(p)$.

(1) Let $M = R^r$, then $M/pM \cong F^r$.

(2) Let $M = R/(a)$ where $a \in R$ nonzero. Then:

$$M/pM \cong \begin{cases} F & \text{if } p|a \text{ in } R \\ 0 & \text{if } \neg p|a \text{ in } R \end{cases}$$

(3) Let $M = R/(a_1) \oplus \dots \oplus R/(a_k)$ where $p|a_i \forall i$, then $M/pM \cong F^k$.

Proof. (1) Consider the natural surjection $R^r \rightarrow (R/(p))^r$. By the First Isomorphism Theorem, (1) follows.

(2) Consider $\pi : R \rightarrow R/(a)$. Observe that $p(R/(a)) = \pi((p))$. Hence by the Lattice Theorem, $p(R/(a)) \cong \frac{(p)+(a)}{(a)}$. Using the fact that the ideal $(p) + (a)$ is generated by a gcd of p and a , the result follows.

(3) Consider the map:

$$M \longrightarrow \underbrace{R/(p) \oplus \dots \oplus R/(p)}_{\#k \text{ times}}$$

Using (2) as a fact, it follows from the First Isomorphism Theorem, as in the proof of (1) of Theorem A.10. □

Theorem A.13. Let R be a P.I.D. and M_1 and M_2 be two finitely-generated R -modules, then:

(1) $M_1 \cong M_2$ iff they have the same free rank and the same list of invariant factors,

(2) $M_1 \cong M_2$ iff they have the same free rank and the same list of elementary divisors.

Proof. In both (1) and (2), the reverse implications are clear.

Suppose $M_1 \cong M_2$, then $\text{Tor } M_1 \cong \text{Tor}(M_2)$ as torsions are mapped isomorphically. Then $R^{r_1} \cong M_1/\text{Tor } M_1 \cong M_2/\text{Tor } M_2 \cong R^{r_2}$, where r_1, r_2 are the free ranks of the torsion-free quotients. Given any nonzero prime $p \in R$, we have $R^{r_1} \cong R^{r_2} \implies R^{r_1}/pR^{r_1} \cong$

$R^{r_2}/_p R^{r_2}$. Then $F^{r_1} \cong F^{r_2}$, by the previous Lemma, where $F \cong R/_p R$. By isomorphism of vector spaces, we have $r_1 = r_2$.

To show the sameness of factors and divisors, it is sufficient to work with torsions, thus we can suppose M_1 and M_2 are torsion R -modules.

It further suffices to prove, for a fixed prime p , if two isomorphic torsion modules of which both have annihilator a power of p , then they shall have the same elementary divisors.

Proceeding by induction on the power of p in the annihilator of M_1 (also follows, of M_2), if p has 0 power, then M_1 and M_2 are both 0 as they are torsion modules and we are done, so suppose it is nonzero. Suppose elementary divisors of M_1 are given as follows:

$$\underbrace{p, p, \dots, p}_{n \text{ times}}, p^{\alpha_1}, \dots, p^{\alpha_s} \text{ where } 2 \leq \alpha_1 \leq \dots \leq \alpha_s$$

Then we have M is isomorphic to a direct sum of cyclic modules whose annihilators are $(p), (p), \dots, (p), (p^{\alpha_1}), \dots, (p^{\alpha_s})$. Then the submodule pM_1 has elementary divisors $p^{\alpha_1-1}, \dots, p^{\alpha_s-1}$. Hence it is a direct sum of cyclic modules whose annihilators are $(1), \dots, (1), (p^{\alpha_1-1}), \dots, (p^{\alpha_s-1})$. Let the elementary divisors of M_2 be given as:

$$\underbrace{p, p, \dots, p}_{n \text{ times}}, p^{\beta_1}, \dots, p^{\beta_t} \text{ where } 2 \leq \beta_1 \leq \dots \leq \beta_t$$

Same argument follows for pM_2 . Then as $M_1 \cong M_2$, we also have $pM_1 \cong pM_2$. By induction, pM_1 and pM_2 have the same elementary divisors, i.e. $s = t$. And using Lemma A.12, we can pass to field by $M_1/_p M_1 \cong M_2/_p M_2$, i.e. $F^{m+s} \cong F^{n+t}$, thus $m = n$, i.e. same elementary divisors.

Having the same invariant factors follows immediately from having the same elementary divisors when we pass from invariant factors to elementary divisors by taking the prime power factors of invariants. $\square$

Corollary A.14 (Fundamental Theorem of Finitely Generated Abelian Groups). *See Theorem 5.1.*

Proof. Take $R = \mathbb{Z}$ in Theorems A.10, A.11, and A.13. $\square$

A.2 Some Category Theory

We are including the definitions of initial and terminal objects, products and co-products; to better motivate pullbacks and pushouts for the refined definition of a CW-complex. One other definition we include is for the natural transformations, needed in the definition of a (formal) homology theory (3.1). This subsection is from [Le14].

Definition A.10. Let $\mathcal{C}$ be a category. Then an object $T \in \mathcal{C}$ is **terminal** in $\mathcal{C}$ if for each object $X \in \mathcal{C}$, there exists a unique morphism $X \xrightarrow{!} T$.

Remark A.5. The dual of a terminal object is an **initial** object. That is, a terminal object is initial in the opposite category.

Definition A.11. Let $\mathcal{C}$ be a category. Given two objects A and B in $\mathcal{C}$, the **product** of A and B is an object denoted $A \times B \in \mathcal{C}$, equipped with two projection morphisms p and q , such that we have the following diagram:

$$\begin{array}{ccc} A \times B & \xrightarrow{q} & B \\ \downarrow p & & \\ A & & \end{array}$$

and the product **universal** with respect to the diagram above. In other words, given any other triple (V, s, t) such that $s : V \rightarrow A$ and $t : V \rightarrow B$, the following diagram commutes with a unique map h :

$$\begin{array}{ccccc} V & & & & \\ & \searrow t & & & \\ & & A \times B & \xrightarrow{q} & B \\ & \swarrow s & \downarrow p & & \\ & & A & & \end{array}$$

$\exists ! h$ (dashed arrow from V to $A \times B$)

Remark A.6. The dual of a product is the **coproduct**.

Example A.1. In Set , the product of two sets A, B is the Cartesian product:

$$A \times B := \{(a, b) \mid a \in A, b \in B\}$$

Example A.2. In $\mathbf{Set}$, the coproduct of two sets A, B is the disjoint union:

$$A \sqcup B := (A \times \{0\}) \cup (B \times \{1\})$$

Definition A.12. Consider the following setting in a category $\mathcal{C}$. Given a ‘corner’; that is, two objects A, B going into another object C with morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$, i.e.

$$\begin{array}{ccc} & B & \\ & \downarrow g & \\ A & \xrightarrow{f} & C \end{array}$$

a **pullback of the morphisms f and g** is a triple (V, p, q) , with the object V and morphisms p, q such that the following diagram commutes,

$$\begin{array}{ccc} V & \xrightarrow{q} & B \\ \downarrow p & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

and the triple (V, p, q) is universal with respect to this diagram, in other words, the following diagram commutes for any other such triple (Q, s, t) :

$$\begin{array}{ccccc} Q & & & & \\ & \searrow \exists! h & & \searrow t & \\ & V & \xrightarrow{q} & B & \\ & \downarrow p & & \downarrow g & \\ & A & \xrightarrow{f} & C & \\ & \swarrow s & & & \end{array}$$

Remark A.7. The dual of a pullback is a **pushout**.

Example A.3. Given a corner $(A, C, B, f : A \rightarrow C, g : B \rightarrow C)$ in $\mathbf{Set}$, the pullback of f and g is a subset of the Cartesian product, given by:

$$V := \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

Example A.4. Given a corner $(A, C, B, f : A \rightarrow C, g : B \rightarrow C)$ in $\mathbf{Set}$, the pushout of f and g is a subset of the disjoint union, given by:

$$W := A \sqcup B / \sim \text{ generated by } f(c) \sim g(c)$$

Definition A.13. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $\mathcal{F}$ and $\mathcal{G}$ be functors $\mathcal{A} \rightarrow \mathcal{B}$. A **natural transformation** $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ is a family of maps $\left(\mathcal{F}(A) \xrightarrow{\alpha_A} \mathcal{G}(A) \right)_{A \in \mathcal{A}}$ in $\mathcal{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathcal{A}$, the following square commutes:

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(A') \\ \alpha_A \downarrow & & \downarrow \alpha_{A'} \\ \mathcal{G}(A) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(A') \end{array}$$

A.3 Further on CW-Complexes

Definition A.14 (Adapted from [Ma99], p. 73, and [Dio8], p. 203). Let X be a topological space, $A \subseteq X$ be a Hausdorff space. A **relative CW-complex structure** on (X, A) consists of:

- (i) A sequence $(X_n)_{n=-1}^{\infty}$ of subspaces of X , such that:

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \text{ where } X = \bigcup_{n=-1}^{\infty} X_n$$

- (ii) A collection $\{e_i^n\}_{i \in I_n}$ of subsets of X_n for each $n \geq 0$, with the following properties:

- (a) X has the **weak topology** w.r.t. $(X_n)_{n=-1}^{\infty}$
- (b) $\forall n \geq 0, \exists \{q_i^n\}_{i \in I_n}$ s.t. $q_i^n : \mathbb{S}^{n-1} \rightarrow X_{n-1}$ is continuous for all $i \in I_n$, and $\forall n \geq 0, \exists \{Q_i^n\}_{i \in I_n}$ s.t. $Q_i^n : \mathbb{D}^n \rightarrow X_n$ is continuous for all $i \in I_n$, with the left diagram below being a *pushout* of the right:

$$\begin{array}{ccc} \coprod_{i \in I_n} \mathbb{S}_i^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} \mathbb{D}_i^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X^n \end{array} \quad \begin{array}{ccc} \coprod_{i \in I_n} \mathbb{S}_i^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} \mathbb{D}_i^n & & \end{array}$$

In other words, $X \cong X_{n-1} \cup_{(\sqcup_{i \in I_n} q_i^n)} \coprod_{i \in I_n} \mathbb{D}_i^n$ and **n-dimensional open cells of X_n** are $e_i = Q_i^n(\text{int}(\mathbb{D}^n))$, closed cells are $\overline{e_i^n} = Q_i^n(\mathbb{D}^n)$, where the image of the $n - 1$ -dimensional circle corresponds to the boundary, $Q_i^n(\mathbb{S}^{n-1}) = \partial(e_i^n)$.

The index sets $\{I_n\}_{n \geq 0}$ determine the cardinality of n-cells $\{e_i^n\}$ for each X_n . $\{q_i^n\}_{i \in I_n}$ are called **attaching maps**. $\{Q_i^n\}_{i \in I_n}$ are called **characteristic maps**. For each $n \geq -1$, X_n is called the **n th-skeleton of (X, A)** . In the case $A = \emptyset$, A is ignored notationally and we have a **CW-complex structure** on X . If $X = X_n$, $X \neq X_{n-1}$, we denote by $n = \dim(X, A)$ the **(cellular) dimension** of (X, A) .

Example A.5. Let $A = \emptyset$, and the index sets be given as $I_0 = I_1 = \{1, 2\}$, $I_2 = \{1\}$ and $I_n = \emptyset$ for $n \geq 3$. Then we have:

$$\{e_1^0, e_2^0\} = X_0 \subset X_1 \subset X_2 \dots$$

with $X = \bigcup_{i=0}^2 X_i$. Given the 1st characteristic maps:

$$q_i^1 : \mathbb{S}^0 \rightarrow X^0; \quad -1 \mapsto e_1^0, \quad 1 \mapsto e_2^0$$

We obtain a CW-complex structure on $\mathbb{S}^1$ shown by the following pushout square on the left:

$$\begin{array}{ccc} \mathbb{S}^0 \sqcup \mathbb{S}^0 & \xrightarrow{q_i^1} & X_0 = \{e_1^0, e_2^0\} \\ \downarrow & & \downarrow \\ \mathbb{D}^1 \sqcup \mathbb{D}^1 & \longrightarrow & X_1 \end{array} \qquad \begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{id} & X_1 = \mathbb{S}^1 \\ \downarrow & & \downarrow \\ \mathbb{D}^2 & \longrightarrow & X_2 = \mathbb{D}^2 \end{array}$$

The pushout square on the right represents attaching a 2-cell to $X_1 = \mathbb{S}^1$, to obtain a CW-structure on $\mathbb{D}^2$, given the 2nd attaching map is the identity. Observe that these two pushout squares can be shown on the same diagram as in Definition A.14.

The following results are from [Br97], pp. 195-196.

Proposition A.15. *Let K be a CW-complex, the following is true:*

1. *If $A \subset K$ has no two points in the same open cell, then A is closed and discrete,*
2. *If $C \subset K$ is compact, then C is contained in a finite union of open cells,*
3. *Each cell of K is contained in a finite subcomplex of K .*

Proof. We first show equivalence between the statements, then prove them.

(1) $\implies$ (2). Given a $C \subset K$ subset, let $A \subset C$ be $A = \{t_i\}_i$ where $t_i \in T_i$ and $\{T_i\}$ are the open sets C touches. Since now A is closed and discrete, it is compact and finite.

(2) $\implies$ (3). Let σ be a cell. Let C be the union of closed cells K_σ and smaller dimensional closed cells. By (2), K_σ is contained in a finite union of open cells of smaller dimension (except U_σ). Take these open sets, for K_σ and for smaller closed cells, to obtain a finite subcomplex.

(2) $\implies$ (3). Consider the (nonempty) intersection $A \cap K_\sigma$ where K_σ is some closed cell.

By (3), the intersection is in some finite subcomplex. Since A has at most one point in common with any open cell, $A \cap K_\sigma$ is finite, thus closed by Proposition 4.1. Since for any $x \in A$, $A - \{x\}$ is also closed, $\{x\}$ is open, and thus A discrete.

We now do induction over skeleta. It's clear (1) holds for $K^{(0)}$, suppose (1) holds for $K^{(n)}$. Then (2) holds for $K^{(2)}$. And since when proving (3) for a particular k -cell, we only used (2) on the subsets of the $K^{(k-1)}$, thus (3) holds for $K^{(n+1)}$. Then (1) holds for $K^{(n+1)}$. We have (1), (2), and (3) holding for all $K^{(n)}$ and as any cell in some skeleton, we have (3), thus also (1) and (2) for K . $\square$

Theorem A.16. If K is a CW-complex, then any compact subset of K is contained in a finite subcomplex.

Proof. Proof immediately follows by (2) and (3) of the previous proposition. $\square$

A.4 Δ -complexes and Simplicial Homology

This subsection is included for the sake of having the intuitive theory of simplicial homology and to have some computations of homology groups at hand. For a more detailed account for this subsection, the reader is referred to [Ha02].

Remark A.8. For Δ -complexes, the following convention is used: the vertices of any subsimplex (spanned by a subset of the vertices of the larger simplex) are ordered with respect to their order in the larger simplex, is used.

Definition A.15. The union of all faces of Δ_p is the **boundary of Δ_p** , denoted $\partial\Delta_p$. The **open simplex $\mathring{\Delta}_p$** is the set difference $\Delta_p - \partial\Delta_p$, the interior of Δ_p .

Definition A.16. A Δ -complex structure is a collection of maps $\sigma_\alpha : \Delta_n \rightarrow X$, where n depends on the index α , such that the following are satisfied:

- (i) The restriction $\sigma_\alpha|_{\dot{\Delta}_p}$ is injective and for each $x \in X$, $x \in \text{im}(\sigma_\alpha|_{\dot{\Delta}_p})$ for only one restriction,
- (ii) For a face F_i^n of Δ_n , the restriction σ_α to F_i^n is already a map $\sigma_\beta : \Delta_{n-1} \rightarrow X$, i.e. $\sigma_\alpha \circ F_i^n = \sigma_\beta$,
- (iii) A set $A \subseteq X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ_n for all σ_α .

A space X with a Δ -complex structure on it is called a **Δ -complex**.

Definition A.17. Define $\Delta_n(X)$ to be the free abelian group with basis as the open simplices $e_\alpha^n = \sigma_\alpha(\dot{\Delta}_n)$ of X . It is called the simplicial n th chain group of X .

Remark A.9. Observe that equipping a Δ -complex with the topology given in (iii) of its definition, the purely combinatorial description of a Δ -complex can be reconsidered as a disjoint union of simplices under the identifications provided by characteristic maps as σ_α 's and attaching maps as restrictions of these characteristic maps to faces of Δ_n 's. Thus a Δ -complex is a CW-complex with more strict maps as simplices.

Definition A.18. Define the **boundary homomorphism** on a Δ -complex X , by defining it on the basis elements:

$$\partial_n : \Delta_n(X) \rightarrow \Delta_{n-1}(X); \quad \partial(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Proposition A.17. $\partial_{n-1} \circ \partial_n = 0$.

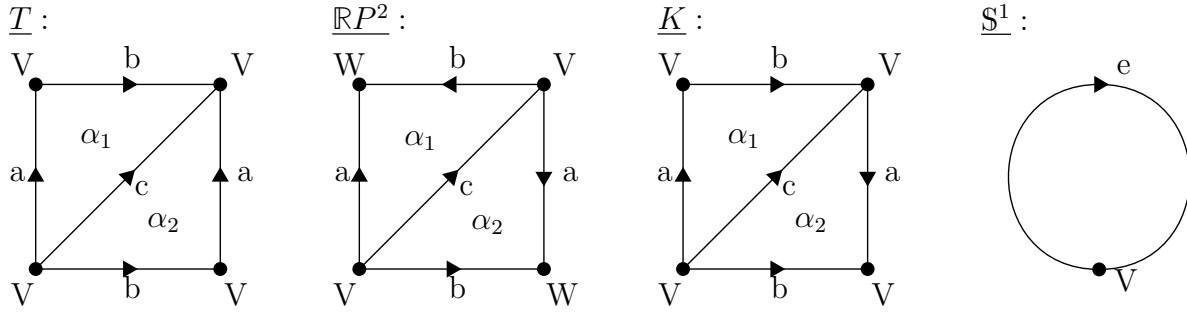
Proof. Proof is done for the singular homology case. □

Definition A.19. For a given boundary homomorphism, define the **n th (simplicial) homology group** $H_n^\Delta(X)$ of X as:

$$H_n^\Delta(X) = \ker \partial_n / \text{im } \partial_{n+1}$$

Example A.6 (Computing simplicial homology groups of some complexes). The following are diagrams for some Δ -structures⁹:

⁹It could be seen from the diagram of T , that we have index sets $I_0 = \{1\}$, $I_1 = \{1, 2, 3\}$, $I_2 = \{1, 2\}$,

Figure 6: Δ -structures on T , $\mathbb{R}P^2$, K , and $\mathbb{S}^1$

Computing the homology groups of the $\mathbb{S}^1$ first, we start with chain groups $\Delta_0(X) \cong \Delta_1(X) \cong \mathbb{Z}$. Since $\partial_1(e) = V - V = 0$, $\ker \partial_1 \cong \mathbb{Z}$ and $\text{im } \partial_1 = 0$ with $\ker \partial_0 \cong \mathbb{Z}$ by definition, and since we have no simplices in dimensions higher than two, we have:

$$H_n^\Delta(\mathbb{S}^1) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, 1; \\ 0 & \text{for } n \geq 2. \end{cases}$$

For the torus T , we have the chain groups $\Delta_0(T) \cong \mathbb{Z}$, $\Delta_1(T) \cong \mathbb{Z}^3$, $\Delta_2(T) \cong \mathbb{Z}^2$, and $\Delta_n(T) = 0$ for $n \geq 3$. We have the following boundary homomorphisms:

$$0 = \Delta_3(T) \xrightarrow{\partial_3=0} \Delta_2(T) \xrightarrow[\alpha_1, \alpha_2 \mapsto a+b-c]{\partial_2} \Delta_1(T) \xrightarrow[a, b, c \mapsto 0]{\partial_1} \Delta_0(T) \xrightarrow{\partial_0=0} 0$$

The homology groups follow as:

$$H_n^\Delta(T) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1, \\ \mathbb{Z} & \text{for } n = 2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

For the real projective plane $\mathbb{R}P^2$, we have chain groups $\Delta_0(\mathbb{R}P^2) \cong \mathbb{Z}^2$, $\Delta_1(\mathbb{R}P^2) \cong \mathbb{Z}^3$, $\Delta_2(\mathbb{R}P^2) \cong \mathbb{Z}^2$, and $\Delta_n(\mathbb{R}P^2) = 0$ for $n \geq 3$. With the remark that $\ker \partial_2 = 0$ (can be

and thus six maps $\sigma_{\alpha_i \in I_i} : \Delta_i \rightarrow T$, where $i = 0, 1, 2$. It could be verified directly from the diagrams that these maps satisfy the conditions for being a Δ -structure.

seen from its matrix representation), the homology groups are calculated as:

$$H_n^\Delta(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}_2 & \text{for } n = 1, \\ 0 & \text{for } n \geq 2. \end{cases}$$

Computed the same way, the klein bottle K has the homology groups:

$$H_n^\Delta(K) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 1, \\ 0 & \text{for } n \geq 2. \end{cases}$$

Example A.7 (Homology groups of $\mathbb{S}^2$). Consider the following complexes on $\mathbb{S}^2$:

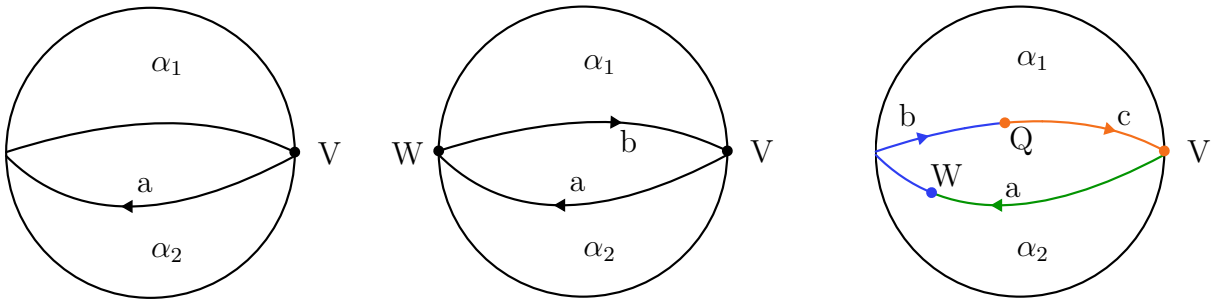


Figure 7: Three complexes on $\mathbb{S}^2$

Two on the left are not Δ -complexes, the rightmost is one. We could have taken the tetrahedron also but reducing the number of simplices simplifies the computation.

We have the chain groups as $\Delta_0(\mathbb{S}^2) \cong \mathbb{Z}^3$, $\Delta_1(\mathbb{S}^2) \cong \mathbb{Z}^3$, and $\Delta_2(\mathbb{S}^2) \cong \mathbb{Z}^2$. We have the following boundary homomorphisms:

$$0 \xrightarrow{\partial_3=0} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \xrightarrow{\partial_0=0} 0$$

where $\partial_1 : a \mapsto W - V$, $b \mapsto Q - W$, $c \mapsto V - Q$ and $\partial_2 : \alpha_1, \alpha_2 \mapsto a + b + c$.

For $H_0^\Delta(\mathbb{S}^2) = \ker \partial_0 / \text{im } \partial_1$ we need to understand ∂_1 . Looking at the complex, we have $H_0^\Delta(\mathbb{S}^2) \cong \langle V, W, Q \rangle / \langle W - V, Q - W, V - Q \rangle$. Observe that $-(W - V) - (Q - W) = V - Q$, hence $V - Q$ is a $\mathbb{Z}$ -linear combination of the other two basis elements. It can be seen those two basis elements are linearly independent, hence we have $H_0^\Delta(\mathbb{S}^2) \cong \mathbb{Z}^3 / \mathbb{Z}^2 \cong \mathbb{Z}$.

For $H_1^\Delta(\mathbb{S}^2)$, by the explanation above, $\ker \partial_1 \cong \mathbb{Z}$, and we have $\operatorname{im} \partial_2 = \langle \alpha_1 - \alpha_2 \rangle \cong \mathbb{Z}$, thus $H_1^\Delta(\mathbb{S}^2) \cong 0$.

For $H_2^\Delta(\mathbb{S}^2)$, it is computed directly as $H_2^\Delta(\mathbb{S}^2) \cong \mathbb{Z}$. For $n \geq 3$, we have $H_n^\Delta(\mathbb{S}^2) \cong 0$.

Thus, in total:

$$H_n^\Delta(\mathbb{S}^2) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ 0 & \text{for } n = 1, \\ \mathbb{Z} & \text{for } n = 2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

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