



Dold-Kan Correspondence

Emir Yusuf Akan

Supervisor: Özgün Ünlü

Course: MATH 492

Semester: Spring 2026

Contents

1	Introduction	2
2	Simplicial Objects	3
2.1	Simplicial Sets	3
2.2	Simplicial Objects	6
2.3	Realization	10
2.4	Kan Complexes	12
3	$\pi_n^{\text{Simplicial}}$ and Simplicial Homotopy	14
3.1	Products and Homotopy of Simplicial Maps	17
4	On $\text{Ch}_{\geq 0}(\text{Ab})$	18
5	Dold-Kan	21
5.1	Proof of Category Equivalence	21
5.2	Proof of Additional Statements	37
6	Application: Eilenberg-MacLane Spaces $K(G, n)$	38
6.1	Further on Realization	39
6.2	Definition of $K(G, n)$	40
6.3	Proof of Existence of $K(G, n)$ in Top	41
6.4	On Uniqueness of $K(G, n)$ up to Homotopy Equivalence	41
A	Appendices	44
A.1	Category Theoretic Notions	44
A.2	Proofs of Well-Definition of $\mathcal{C}$ and $\mathcal{D}$	45
A.3	A Motivating Example for Realization	47
	References	49

Abstract

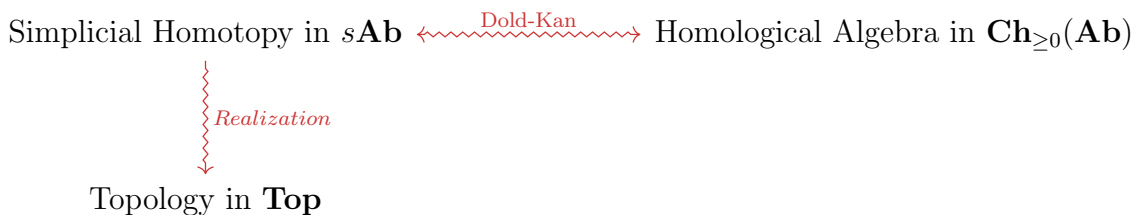
In this project, we first introduce the prerequisites from simplicial homotopy theory and homological algebra. We then study the proof of Dold-Kan correspondence, explicitly constructing the functors $\mathcal{N} : s\mathbf{Ab} \rightleftarrows \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) : \mathcal{K}$ and establishing category equivalence between simplicial abelian groups and non-negatively graded chain complexes. After studying some important consequences of Dold-Kan, we explore how this machinery is applied directly to topology through Eilenberg-MacLane spaces $K(G, n)$. Specifically, we study the proof of their existence and, to some extent, explore the arguments for their uniqueness up to homotopy equivalence.

1 Introduction

In broad terms, algebraic topology tries to answer topological problems via extracting algebraic invariants and studying their properties instead. A central framework for this translation is homotopy theory, which classifies spaces up to continuous deformation. A combinatorial approach to this is simplicial homotopy theory, where one works with discrete simplicial objects. When these objects possess some underlying abelian structure, such as with simplicial abelian groups, a brilliant bridge emerges between simplicial homotopy theory and homological algebra. This bridge is provided by the Dold-Kan correspondence.

Through Dold-Kan correspondence some homotopical problems can be approached directly by homological algebra. One example is the construction of Eilenberg-MacLane spaces $K(G, n)$. Their construction via topological methods is cumbersome even in really low dimensions. Luckily, Dold-Kan allows us to construct them in any dimension easily.

The first part of the project focuses on the bridge between simplicial homotopy and homological algebra, established via Dold-Kan. And the second part of the project focuses on connections to topology through the concept of realization as the main machinery.



2 Simplicial Objects

This section introduces necessary prerequisites for one of the categories in Dold-Kan, simplicial abelian groups $s\mathbf{Ab}$. The goal is to motivate and provide sufficient simplicial homotopy theory, on $s\mathbf{Set}$.

2.1 Simplicial Sets

To really understand the necessity and intuition of simplicial sets, it is imperative to go through many layers of abstractions that motivate its purely combinatorial definition. We refer the reader to the two main sources they can utilize to fill that background, [MKP25] and [Fri23]. This section is an adaptation of selected parts of the latter reference.

Definition 2.1. A **simplicial set** consists of a sequence of sets $X_0, X_1, \dots$ and for each i with $0 \leq i \leq n$, **face maps** $d_i : X_n \rightarrow X_{n-1}$ for all $n \geq 1$, with **degeneracy maps** $s_i : X_n \rightarrow X_{n+1}$ for all $n \geq 0$ such that they satisfy the **simplicial identities**:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i & \text{if } i < j, \\ d_i s_j &= s_{j-1} d_i & \text{if } i < j, \\ d_j s_j &= d_{j+1} s_j = \text{id}_{X_n}, \\ d_i s_j &= s_j d_{i-1} & \text{if } i > j + 1, \\ s_i s_j &= s_{j+1} s_i & \text{if } i \leq j. \end{aligned}$$

Definition 2.2. A **degenerate n -simplex** in a simplicial set X is an element $x \in X_n$ s.t. it can be written as $s_i y$ for some $y \in X_{n-1}$ and any i . If a simplex is not degenerate, it is called **nondegenerate**.

Example 2.1. Recall ordered simplicial complexes (see [MKP25], §1-3). For example, the ordered 1-simplex $|\Delta^1| = [0, 1]$ is given by the vertices $X_0 = \{0, 1\}$ and the only edge $X_1 = \{[0, 1]\}$ in abstract notation.

An example to a simplicial set would be the ones that we can obtain from simplicial complexes by way of “adjoining all degeneracies to it”: given a simplicial complex $|X|$, let X be the collection of sets $\{X_n\}_{n \geq 0}$ where X_n is set of all simplices $[v_{i_0}, \dots, v_{i_n}]$ in which $v_{i_k} \leq v_{i_{k+1}}$, and the set of vertices $\{v_{i_0}, \dots, v_{i_n}\}$ spans a simplex of $|X|$. We have

the face and degeneracy maps defined¹ as follows:

$$\begin{aligned} d_j[v_0, \dots, v_m] &= [v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_m] \\ s_j[v_0, \dots, v_m] &= [v_0, \dots, v_j, v_j, \dots, v_m] \end{aligned} \tag{1}$$

In the case of $|\Delta^1|$, notice that the maps d_k that omit a vertex were already conceivable but the novel thing is provided with maps s_k that repeat a vertex. Roughly speaking, we adjoin *hidden* n -simplices to the basic line segment we know (see Figure 1).

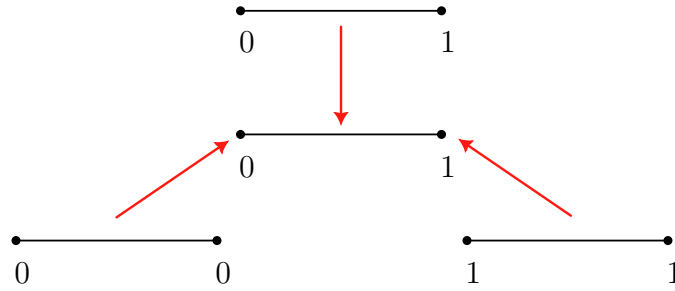


Figure 1: What it means to adjoin degenerate 1-simplices in the case of $|\Delta^1|$

Hence, we obtain the standard ordered 1-simplex Δ^1 , given as:

$$\Delta^1 = \left\{ X_0 = \{0, 1\}, X_1 = \{[0, 0], [0, 1], [1, 1]\}, \bigcup_{n \geq 2} X_n \right\}$$

In the more general case of Δ^n , the described maps can be shown to satisfy the simplicial identities by definition, so that Δ^n is a valid simplicial set.

Definition 2.3. The simplicial set associated to $|\Delta^n|$ is referred to as **the standard ordered n -simplex**, denoted Δ^n . We shall use the finite ordered set notation $[0, \dots, n]$ for its elements adapted from simplicial complexes; its maps d_k and s_k are given as in Equation (1).

Example 2.2. Consider $|\Delta^2|$ and its associated simplicial set Δ^2 . As an example to how face and degeneracy maps behave in this case, the second identity in Definition 2.1 is visualised below with faces and degeneracies defined as in (1), the other identities can be seen similarly:

¹Here, the square bracket notation, that denotes a sequence of outputs, is used for convenience.

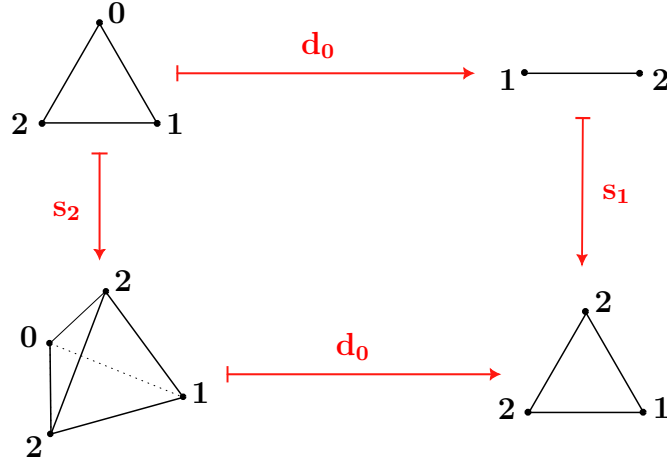


Figure 2: $d_i s_j = s_{j-1} d_i$ if $i < j$ visualised for $j = 2, i = 0$; infinitely many degenerate simplices are not drawn for obvious reasons.

Example 2.3. The standard 0-simplex Δ^0 consists of $X_n = \{\underbrace{[0, \dots, 0]}_{n+1 \text{ times}}\}$ for each $n \geq 0$ with unique face and degeneracy maps.

Definition 2.4. An important simplicial set is the **singular set** $\mathcal{S}(X)$ of a topological space X . Given a space X , let:

$$\mathcal{S}_n(X) := \text{Hom}_{\mathbf{Top}}(|\Delta^n|, X)$$

where face and degeneracy maps for each $0 \leq i \leq n$ are defined as:

$$\begin{aligned} d_i: \mathcal{S}_n(X) &\rightarrow \mathcal{S}_{n-1}(X) & \text{and} & & s_i: \mathcal{S}_{n+1}(X) &\rightarrow \mathcal{S}_n(X) \\ d_i \sigma &\mapsto \sigma \circ \delta^i & & & s_i \sigma &\mapsto \sigma \circ \eta^i \end{aligned}$$

Here, the map $\delta^i: |\Delta^{n-1}| \rightarrow |\Delta^n|$ is the simplicial inclusion determined by the vertex assignment $[0, \dots, n-1] \mapsto [0, \dots, \hat{i}, \dots, n]$ where hat denotes omission; and the map $\eta^i: |\Delta^{n+1}| \rightarrow |\Delta^n|$ is the simplicial collapse determined by the vertex assignment $[0, \dots, n+1] \mapsto [0, \dots, i, i, \dots, n]$. *C.f.* Example 2.5 and Remark 2.3 to see the exact nature of these maps.

Example 2.4. A degenerate singular 3-simplex example would be the following where δ is a simplicial collapse and σ is a non-collapsed singular 2-simplex:

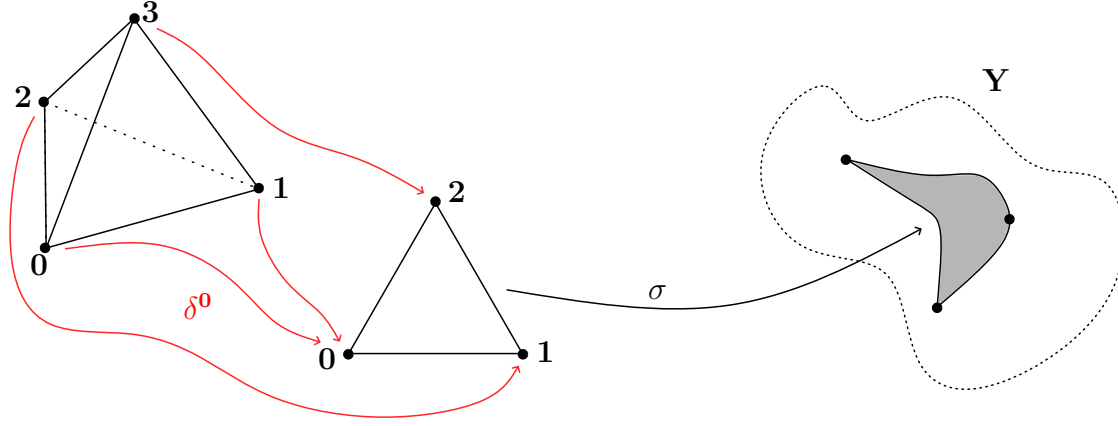


Figure 3: A degenerate singular simplex

Proof of $\mathcal{S}(X)$ being a simplicial set. This proof uses the fact that $\{|\Delta^n|\}(-)$ is a co-variant functor $\mathbf{\Delta} \rightarrow \mathbf{Top}$ (see Example 2.5). Then the singular set functor rewritten as:

$$\mathcal{S}_*(X) = \text{Hom}_{\mathbf{Top}}(\{|\Delta^n|\}(-), X)$$

precomposes the singular simplices with cofaces and codegeneracies of $\{|\Delta^n|\}$. Thus, we automatically get simplicial identities by functoriality. $\square$

2.2 Simplicial Objects

In this section, we introduce all relevant definitions in the forms that they are utilized later. We begin by introducing our prototype category on which we define simplicial objects as functors.

Definition 2.5. Let the **simplex category**, denoted $\mathbf{\Delta}$, be the category with objects as finite ordered sets² $[n] = \{1, 2, \dots, n\}$ and morphisms as nondecreasing monotone functions $[m] \rightarrow [n]$.

One can observe that even in the case of $[1] \rightarrow [n]$, we have $\binom{n+2}{2}$ nondecreasing monotone functions, so that the massive total amount of functions in $\mathbf{\Delta}$ can only be imagined. For this, for each n and i with $0 \leq i \leq n$, we introduce **coface** $d^i : [n-1] \rightarrow [n]$

²Be wary that we use square brackets to denote objects as ordered sets in the category $\mathbf{\Delta}$, whereas we have used the square-bracket notation in the simplicial set Δ^n to denote a sequence of outputs.

and **codegeneracy maps** $s^i : [n+1] \rightarrow [n]$ into Δ :

$$d^i(j) = \begin{cases} j, & \text{if } j < i; \\ j+1, & \text{if } j \geq i. \end{cases} \quad \text{and} \quad s^i(j) = \begin{cases} j, & \text{if } j \leq i; \\ j-1, & \text{if } j > i. \end{cases}$$

Here we recall the definition of a simplicial set and dualize it.

Definition 2.6. A **cosimplicial set** consists of a sequence of sets $X^0, X^1, \dots$ and for each i with $0 \leq i \leq n$, **coface maps** $d^i : X_n \rightarrow X_{n-1}$ for all $n \geq 1$, with **codegeneracy maps** $s^i : X_n \rightarrow X_{n+1}$ for all $n \geq 0$ such that they satisfy the **cosimplicial identities**:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} & \text{if } i < j, \\ s^j d^i &= d^i s^{j-1} & \text{if } i < j, \\ s^j d^j &= s^j d^{j+1} = \text{id}_{X_n}, \\ s^j d^i &= d^{i-1} s^j & \text{if } i > j+1, \\ s^j s^i &= s^i s^{j+1} & \text{if } i \leq j. \end{aligned}$$

Proposition 2.1. *The maps d^i and s^i in Δ satisfy the cosimplicial identities.*

Proof. Each identity follows immediately by definition of maps d^i and s^i . $\square$

An **epi** is an epimorphism, hence a surjection; and a **monic** is a monomorphism, hence an injection in the simplex category Δ .

An important lemma that is utilized heavily in the proof of Dold-Kan is the following.

Lemma 2.2. *Let $\alpha : [n] \rightarrow [m]$ be any morphism in Δ . There exists a unique epi-monic factorization $\alpha = DS$ where:*

$$\begin{aligned} D &= d^{i_1} \dots d^{i_s} \text{ with } 0 \leq i_s < \dots < i_1 \leq m \\ S &= s^{j_1} \dots s^{j_t} \text{ with } 0 \leq j_1 < \dots < j_t \leq n \end{aligned}$$

Proof. The proof of this is direct. The reader is referred to the proof given in [Wei94]. $\square$

Corollary 2.3. *The morphisms in Δ are generated uniquely by coface and codegeneracy maps.*

Corollary 2.4. *The opposite simplex category Δ^{op} is generated uniquely by face maps, denoted d_i , and degeneracy maps, denoted s_i ; in which our notation is deliberately consistent with previous examples. These maps satisfy the simplicial identities.*

We have just observed how the categories Δ and Δ^{op} can be *blueprints* for cosimplicial and simplicial identities. And if we were to consider the category **Set**, we get simplicial sets as follows.

Definition 2.7. A simplicial set X is a contravariant functor $X : \Delta \rightarrow \mathbf{Set}$.

Recalling the standard simplicial sets Δ^n (2.3), we see that they are the prototype simplicial sets through the following incredible result.

We realize that Δ^n is the functor given by $\Delta[n] = \text{Hom}_{\Delta}(-, [n])$ where $\Delta[n]$ is a more appropriate notation for this context. More explicitly, it is the following where contravariance is forced by precomposition:

$$\begin{array}{ccc} [m] & \mapsto & \text{Hom}_{\Delta}([m], [n]) \\ d^i \downarrow & \longmapsto & \uparrow \Delta[n](d^i) = - \circ d^i \\ [m+1] & \mapsto & \text{Hom}_{\Delta}([m+1], [n]) \end{array}$$

Proposition 2.5. *For any simplicial set X , its n -simplices $x \in X_n$ are in bijective correspondence with simplicial maps $\Delta[n] \rightarrow X$.*

Proof. By application of Yoneda lemma (A.2), one sees that we have the following bijection:

$$\text{Hom}_{\mathbf{sSet}}(\Delta[n], X) \cong X_n$$

where a simplicial map $f : \Delta[n] \rightarrow X$ determines an n -simplex $x_f \in X_n$ as $x_f = f(\text{id}_{[n]})$ and conversely a simplex x defines a simplicial map f_x defined on the monotonic maps $\lambda \in \text{Hom}_{\Delta}([m], [n])$ by $f_x(\lambda) = X(\lambda)(x) \in X_m$. $\square$

Remark 2.1. Proposition 2.5 tells us that the standard simplices individually correspond to n -simplices for any simplicial set X . They are actually far more important. In fact, the following is true:

Geometrically, any simplicial set X can be constructed by taking a disjoint standard simplex $\Delta[n]$ for every n -simplex $x \in X_n$, and “gluing” them all together along their faces and degeneracies according to the internal combinatorial structure of X .

Or put more concisely:

Simplicial sets are made up of standard simplices.

Unfortunately, exploring why this is the case would detour the project afar. See Proposition A.4 for its formal statement.

The functorial definition of simplicial sets generalizes to any category.

Definition 2.8. A **simplicial object** X in category $\mathcal{C}$ is a contravariant functor $\Delta \rightarrow \mathcal{C}$. Dually, a **cosimplicial object** Y in category $\mathcal{C}$ is a covariant functor $\Delta \rightarrow \mathcal{C}$.

Definition 2.9. A **simplicial (set) morphism** $f : X \rightarrow Y$ is a natural transformation of simplicial sets X, Y as functors. Also called a simplicial map in short.

Definition 2.10. The collection of simplicial sets and simplicial morphisms forms the (functor) category $\mathbf{Set}^{\Delta^{op}}$. We denote this category with $s\mathbf{Set}$. Analogously, the collection of simplicial objects in $\mathcal{A}$ and morphisms forms the category $s\mathcal{A}$.

Definition 2.11. The simplicial abelian groups with face and degeneracy maps as group homomorphisms defined as satisfying simplicial identities forms the category $s\mathbf{Ab}$. This is one of the two points of interest in Dold-Kan. It has objects as simplicial abelian groups $\mathcal{A} : \Delta \rightarrow \mathbf{Ab}$ and morphisms as maps between.

Remark 2.2. One can see that the combinatorial definition of a simplicial set (c.f. 2.1) is equivalent to the functorial definition (c.f. 2.7) by directly translating side-to-side using the definition of Δ (2.5): the sets X_n correspond to images $X([n])$, face and degeneracy maps are already in the same notation, and identities follow from Δ by functoriality. Similarly, the combinatorial definition of a simplicial set generalizes to any simplicial object in a category $\mathcal{C}$, where face and degeneracy maps are realized as morphisms within $\mathcal{C}$.

An important historical remark is that the maps d^i and s^i that we defined in the simplex category Δ are traditionally referred to as “face” and “degeneracy” maps. This is due to the historical origins of the theory of simplicial objects where “face” geometrically refers to the *face inclusion map*.

[Wei94] prefers this and we give the geometric motivating example next. However, modern simplicial homotopy theorists prefer to use the technically correct terminology, see [GJ09].

Example 2.5. Recall the (geometric) n -simplex $|\Delta^n|$ in $\mathbb{R}^{n+1}$. To obtain $\{|\Delta^n|\}_{n \geq 0}$ as a functor of some variance $\Delta \rightarrow \mathbf{Top}$, consider the following. We assign objects as $[n] \mapsto V_{|\Delta^n|}$, i.e. elements of the ordered set $[n]$ are assigned to the vertex set $V_{|\Delta^n|}$ where $i \mapsto v_i = e_{i+1}$ (the standard basis vectors of $\mathbb{R}^{n+1}$). This means, a map $\alpha : [k] \rightarrow [l]$ corresponds to a vertex map of geometric simplices, that sends the vertices of $|\Delta^k|$ to $|\Delta^l|$ to by the rule $\alpha(v_i) = v_{\alpha(i)}$. In particular, if α is a coface map d^j or a codegeneracy map s^j , the rule respectively becomes $d^j(v_i) = v_{d^j(i)}$ and $s^j(v_i) = v_{s^j(i)}$. This vertex map extends linearly to give a continuous map $\alpha_* : |\Delta^k| \rightarrow |\Delta^l|$ so that we have the morphism assignment:

$$\begin{array}{ccc} [k] & & |\Delta^k| \\ \alpha \downarrow & \longmapsto & \downarrow \alpha_* \\ [l] & & |\Delta^l| \end{array}$$

which makes the sequence of spaces $\{|\Delta^n|\}_{n \geq 0}$ into a covariant functor $\{|\Delta^n|\}(-) : \Delta \rightarrow \mathbf{Top}$, i.e. a cosimplicial topological space.

Remark 2.3. In this topological setting of $\{|\Delta^n|\}$, the i th coface map means inclusion of the i th face and the i th codegeneracy map means projection onto the i th face. These maps are exactly the maps given in the singular set example, introduced in Example 2.4

2.3 Realization

In this subsection, we investigate a connection from simplicial sets to topological spaces through simplicial realization. This connection is important for topological applications of the Dold-Kan correspondence, e.g. in the study of Eilenberg-MacLane spaces. See Subsection 6.1 for further discussion.

Definition 2.12. Given a simplicial set X , equip each set X_n with the discrete topology, and consider the standard n -simplex $|\Delta^n|$ with the standard topology and with coface

d^i and codegeneracy maps s^i described in Example 2.5, the **simplicial realization** $|X|$ is given by:

$$|X| := \coprod_{n=0}^{\infty} X_n \times |\Delta^n| / \sim$$

where $\sim$ is the equivalence relation generated by the following relations:

- (i) $(x, d^i p) \sim (d_i x, p)$ for $x \in X_{n+1}$, $p \in |\Delta^n|$,
- (ii) $(x, s^i p) \sim (s_i x, p)$ for $x \in X_{n-1}$, $p \in |\Delta^n|$.

The generating identifications (i) and (ii) in the definition of simplicial realization needs to be elucidated from the geometric point of view. The sketches that follow are inspired from [Fri23].

The face identification (i) is sketched in Figure 4. In the product $\coprod X_n \times |\Delta^n|$, we have piles of disjoint simplices where, now, the abstract simplices “ x ” and “ $d_i x$ ” have topological meaning, precisely as they are now *points* in the x –indexed geometric simplices $x \times |\Delta^{n+1}|$ and $d_i x \times |\Delta^n|$. Naturally, the face simplex $(d_i x \times |\Delta^n|)$ that is floating separately must be identified with the face of the original simplex $(x \times |\Delta^n|)$. Recall from Example 2.5 & Remark 2.3 that d^i corresponds to face inclusion. Exactly with this face inclusion, with a chosen point p in $|\Delta^n|$, the point $(d_i x, p)$ from the face simplex, is identified with the point $(x, d^i p)$ by pulling p into the simplex by d^i .

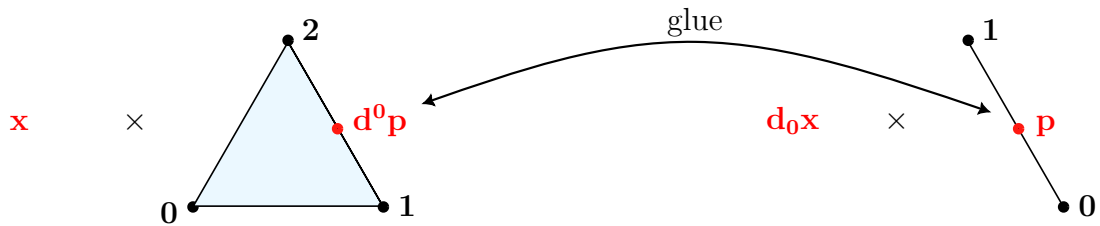


Figure 4: Sketch of the face identification for the case $(x, d^0 p) \sim (d_0 x, p)$ where $x \in X_2$ and $p \in |\Delta^1|$; with $(x, d^0 p) \in x \times |\Delta^2|$ and $(d_0 x, p) \in d_0 x \times |\Delta^1|$

The degeneracy identification (ii) is sketched in Figure 5. Recall from Example 2.5 & Remark 2.3 that s^i corresponds to projection onto the j th face. With this identification, the degenerate n –simplices $s_i x \times |\Delta^i|$ are collapsed into the $(n - 1)$ –simplices they are

degeneracies of, by gluing from the point p considered in the pre-projection n -simplex, to the post-projection $(n - 1)$ -simplex.

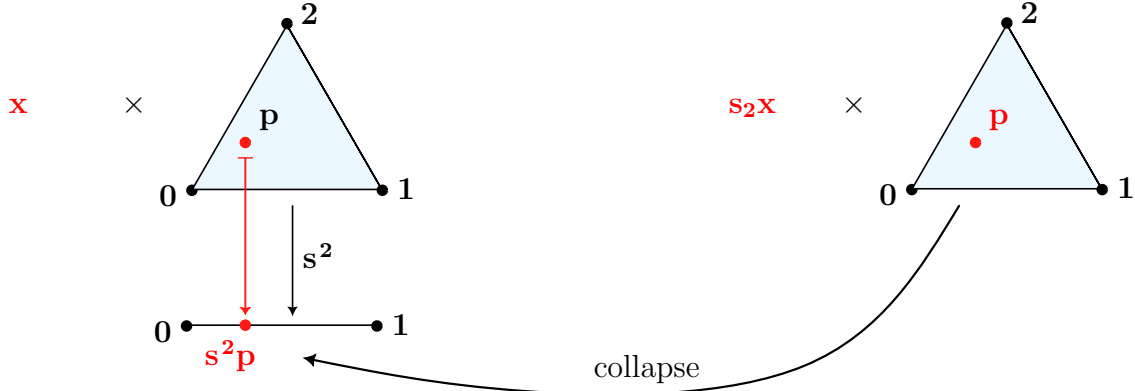


Figure 5: Sketch of the degeneracy identification for the case $(x, s^2p) \sim (s_2x, p)$

Example 2.6. The simplicial realization of Δ^0 (see Example 2.3) is the space with a single point.

2.4 Kan Complexes

Arbitrary simplicial sets do not yield well-defined $\pi_n^{\mathbf{Simplicial}}$, for that, one needs Kan complexes. In this subsection, we motivate them.

Definition 2.13. A simplicial set X is called a **Kan complex** or to be as **fibrant**, if it satisfies the **Kan condition**, i.e. given n and k with $0 \leq k \leq n + 1$:

if the collection of n -simplices $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1} \in X_n$ satisfy the condition $d_i x_j = d_{j-1} x_i$ for all $i < j$ with $i \neq k$ and $j \neq k$, then there exists a $(n + 1)$ -simplex $y \in X_{n+1}$ such that $d_i y = x_i$ for all $i \neq k$.

See [Kan55] for D. Kan's first formulation of the condition where he showed that $\mathcal{S}(X)$ of a space X is fibrant. We do it by expanding upon the proof of [May67].

Proposition 2.6. Given a topological space X , its singular set $\mathcal{S}(X)$ is a Kan complex.

Proof. First, observe that the “if” part of the Kan condition corresponds to specifying a singular simplex $\sigma_i : |\Delta^n| \rightarrow X$ for each (n) -face, $d_i \Delta^{n+1}$ of Δ^{n+1} . The compatibility condition is satisfied by definition and due to this, we can glue all these continuous maps

into one continuous map (see Pasting Lemma, [Mun75]) to end up with

$$f : R \rightarrow X$$

where R is the the boundary of the $(n + 1)$ -simplex with the k -th face removed. Next observation, is the fact that there is a continuous retraction π :

$$\pi : |\Delta^{n+1}| \rightarrow R$$

so that $y := f \circ \pi$ is a well-defined $(n + 1)$ -simplex. The boundary conditions follow by construction so that the “then” part of the Kan condition is satisfied. $\square$

This historically first application of Kan condition provides the motivation for a more intuitive formulation of the condition.

Definition 2.14. Consider the geometric n -simplex $|\Delta^n|$. We define the k th horn Λ_k^n on the n -simplex as the simplicial set associated to the subcomplex $|\Lambda_k^n|$ of $|\Delta^n|$ obtained by removing $\text{int}(|\Delta^n|)$ and $\text{int}(d_k|\Delta^n|)$. In other words, it is:

$$\Lambda_k^n := \{[i_0, \dots, i_m] \mid 0 \leq i_0 \leq \dots \leq i_m \leq n, m \in \mathbb{N}\}$$

subject to the conditions that we can't have $[0, \dots, n] \in \Lambda_k^n$ nor $[0, \dots, k-1, k+1, \dots, n] \in \Lambda_k^n$.

Example 2.7. The standard example is on $|\Delta^2|$, one can easily understand the associated simplicial sets, i.e. horns, from the following realization sketch:

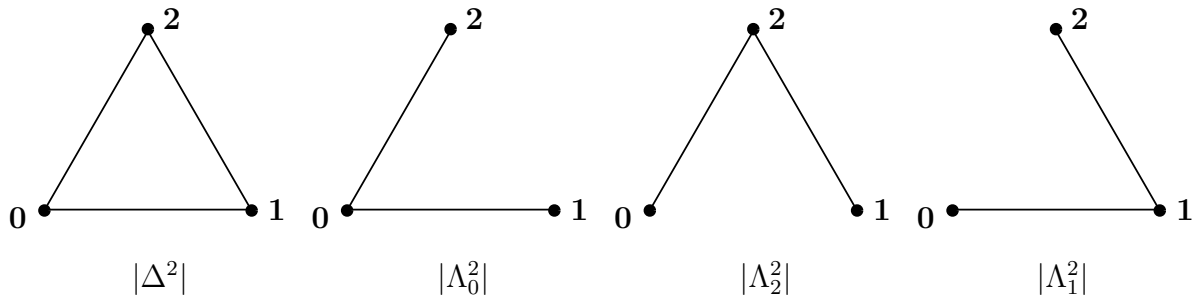


Figure 6: Horns on $|\Delta^2|$

We can now present the equivalent formulation of the Kan condition. The equivalence of definitions follows directly from the representability remark at Proposition 2.5.

Definition 2.15. The simplicial object X satisfies the **Kan condition** (also, horn-filling condition) if any morphism of simplicial sets $\Lambda_k^n \rightarrow X$ can be extended to a morphism $\Delta^n \rightarrow X$.

The horn part of this definition corresponds to the “if” of Definition 2.13, and the extension part corresponds to the “then”.

Example 2.8. No n -simplex Δ^n for $n \geq 1$ satisfies the Kan condition. Consider the map $\Lambda_0^2 \rightarrow \Delta^1$ given by $[0, 1] \mapsto [0, 1]$ and $[0, 2] \mapsto [0, 0]$. The assignment is a valid simplicial map as in the case of maps into a standard simplex, the naturality condition translates to the condition that the vertex map must be non-strictly order-preserving (of simplices that exist in the domain). And this is satisfied: $0 \leq 2 \implies f(0) = 0 \leq 0 = f(2)$ and $0 \leq 1 \implies f(0) = 0 \leq 1 = f(1)$. However, any extension to Δ^2 will remove the order preservation. For the same reason of being able to send into degenerate simplices, no Δ^m for $m \geq 1$ is a Kan complex.

One might ask the question: For each $n \geq 0$, we have $n + 1$ horns, and on each horn, we can have infinitely many simplicial maps into a simplicial set X , so how does one even consistently find simplicial sets to work with? Thankfully, there are simplicial sets with extra structures that automatically allow them to satisfy the Kan condition. One example was already shown, $\mathcal{S}(X)$ in Proposition 2.6; another is simplicial groups. The first result, originally on simplicial abelian groups, is due to [Moo58].

Theorem 2.7 ([Wei94], Lemma 8.2.8). If G is a simplicial group, its underlying simplicial set is a Kan complex.

Corollary 2.8. *The underlying sets of simplicial abelian groups are Kan complexes.*

3 $\pi_n^{\text{Simplicial}}$ and Simplicial Homotopy

In this section, we define $\pi_n^{\text{Simplicial}}$ and homotopy of simplicial maps.

Definition 3.1. Let X be a simplicial set. Two n -simplices $x, x' \in X_n$ are called **homotopic**, denoted $x \sim x'$, if:

- (1) $d_i x = d_i x'$ for all i with $0 \leq i \leq n$,

(2) There exists $y \in X_{n+1}$ such that:

- $d_n y = x$,
- $d_{n+1} y = x'$, and
- $d_i y = s_{n-1} d_i x = s_{n-1} d_i x'$, for all i with $0 \leq i \leq n-1$.

This definition, except the last condition on the homotopy y , can be elucidated using 0-simplices, as shown in the next example. The last condition reads as *faces of y other than x and x' are just the $(n-1)$ th degeneracy of faces of x and x'* .

Example 3.1. A 1-simplex p in a simplicial set X can be called a *path* as equivalently, a 1-simplex, from the representability remark at Proposition 2.5, is a simplicial map $p : \Delta^1 \rightarrow X$. Then $d_1 p = p[0]$ can be called the *initial point* of the path, and $d_0 = p[1]$ the *final point*. And we see that a homotopy of two 0-simplices $a, b \in X_0$ of X from a to b is just a path $p \in X_1$ with initial point a and final point b .

Proposition 3.1 ([May67], Proposition 3.2). *Let K be a Kan complex, then $\sim$ is an equivalence relation on the n -simplices of K for $n \geq 0$.*

Proof. The relation is reflexive because for an n -simplex x , its n th degeneracy is homotopy on itself, i.e. $d_n s_n x = d_{n+1} s_n x$ and $d_i s_n y = s_{n-1} d_i s_n y$ for $0 \leq i \leq n$ via simplicial identities. Symmetry follows from the symmetry of equality “ $=$ ”. For transitivity, let us have $y' : x \sim x'$ and $y'' : x' \sim x''$. The $n+2$, $(n+1)$ -simplices $d_0 s_n s_n x', \dots, d_{n-1} s_n s_n x', y', y''$, skipping $d_i s_n s_n x$ for some $0 \leq i \leq n-1$, satisfies the “if” part of Kan condition. Then by the Kan condition, there exists an $(n+2)$ -simplex z such that $d_i z = d_i s_n s_n x$. The homotopy conditions follow directly from simplicial identities. $\square$

We define a “basepoint” for simplicial sets to be able to define simplicial homotopy groups.

Definition 3.2. Let X be a simplicial set and $*$ $\in X_0$ be a 0-simplex. The element $*$ generates a simplicial set by degeneracies, in X , similar to the standard 0-simplex (c.f. Example 2.3). We denote any simplex in this simplicial set with $*$ by an abuse of notation. If we have a Kan complex K , the pair $(K, *)$ is called a **Kan pair**.

To define simplicial homotopy groups, we need a way to mirror the idea of “the set of all homotopy classes of *loops* at the basepoint” (recall [Hat02], p. 26) from topological

homotopy. To that end, given a Kan pair $(K, *)$, we define:

$$\tilde{K}_n := \{x \in K_n \mid d_i x = *, 0 \leq i \leq n\} = \tilde{K}_n = \bigcap_{i=0}^n \ker(d_i) \quad (2)$$

with the vacuous case $\tilde{K}_0 = K_0$.

Definition 3.3. Given a Kan pair $(K, *)$, we define $\pi_n(K, *)$ as:

$$\pi_n^{\text{Simplicial}}(K, *) = \tilde{K}_n / \sim$$

For the case of simplicial abelian groups, the group structure and being abelian comes by default. For reasons that will be seen in Subsection 6.1, we denote $\pi_n^{\text{Simplicial}}$ by π_n .

Proposition 3.2. *Given a Kan pair $(A, *)$ with an abelian group A , $\pi_n(A, *)$ forms a group where the multiplication comes from A . Moreover, $\pi_n(A, *)$ is abelian for $n \geq 0$.*

Proof. Because the underlying set of A is a Kan complex (see Theorem 2.7), particularly, when restricted to $\tilde{A}_n$, as $\sim$ respects group addition as d_i are $\mathbb{Z}$ -linear, π_n is well-defined. $\square$

For the general case of why, for any Kan pair, $\pi_n(K, *)$ forms a group, one needs to work harder. For example, the fact given for simplicial abelian groups obviously does not work for the Kan pair $\pi_n(\mathcal{S}(X), *)$ given a space X .

Definition 3.4 (Multiplication in $\pi(X, *)$; [Fri23], Definition 9.8). Let X be a Kan complex. For some $n \geq 1$, consider $x, y \in \tilde{X}_n$ such that $d_i x = d_i y$ for all i . Let Δ_n^{n+1} be the horn of Δ^{n+1} in X such that the face corresponding to $d_{n+1}\Delta^{n+1}$ is y , the face corresponding to $d_{n-1}\Delta^{n+1}$ is x , and the faces corresponding to all other sides of the horn are in $*$, i.e. in $\tilde{X}_n$. Let z be an extension of the horn to Δ^{n+1} (provided by the Kan condition). Then define:

$$xy = \text{homotopy class of } d_n z \text{ in } \pi_n(X, *)$$

The following propositions are left without proof as they use the Kan condition extensively to check the conditions, paralleling the proof of Proposition 3.1. For the proofs, see [Fri23] Prop. 9.9, Theorem 9.10, and [May67] Prop. 4.4, respectively.

Proposition 3.3. *The multiplication in Definition 3.4 is a well-defined function $\pi_n(X, *) \times \pi_n(X, *) \rightarrow \pi_n(X, *)$.*

Proposition 3.4. *The multiplication in Definition 3.4 makes $\pi_n(X, *)$ a group for any Kan complex X . Moreover, $\pi_n(X, *)$ is abelian for $n \geq 2$.*

3.1 Products and Homotopy of Simplicial Maps

This subsection is about an additional statement of Dold-Kan, namely on what it means for two simplicial maps to be homotopic.

Definition 3.5. Given X, Y as simplicial objects in some category³, we define the product $X \times Y$ through $(X \times Y)_n := X_n \times Y_n$ where the face and degeneracy maps are defined as follows:

$$d_i(x, y) = (d_i x, d_i y) \quad \text{and} \quad s_i(x, y) = (s_i x, s_i y)$$

We can observe that in the case of simplicial sets, the product is just component-wise Cartesian product of sets.

The following is a synthesis of definitions from [Fri23] and from the definition in Theorem 8.3.12 of [Wei94].

Definition 3.6 (Simplicial homotopy: Definition I). Let $f, g : X \rightarrow Y$ be two simplicial maps between simplicial sets X and Y . f and g are called **(simplicially) homotopic** if there is a simplicial map $H : X \times \Delta^1 \rightarrow Y$ such that $H|_{X \times 0} = f$ and $H|_{X \times 1} = g$.

In other words, if the following diagram commutes:

$$\begin{array}{ccccc} X & \xleftarrow{i_0} & X \times \Delta^1 & \xleftarrow{i_1} & X \\ & \searrow f & \downarrow H & \swarrow g & \\ & & Y & & \end{array}$$

The reason for introducing this definition from the product perspective is that it right away reveals that we mirror topological homotopy. However, for until the end of Dold-Kan, an equivalent combinatorial definition of homotopy is used. The actual use of the homotopy from the product-perspective begins with Eilenberg-MacLane spaces (see Subsection 6.1). For equivalence of definitions see Theorem 8.3.12 of [Wei94].

³To be precise, we need the category to have products, i.e. the product in $X_n \times Y_n$ is the product functor internal to the category on which X and Y are defined.

Definition 3.7 (Simplicial homotopy: Definition II). Let $f, g : X \rightarrow Y$ be two simplicial maps on simplicial sets X and Y . f and g are called **(simplicially) homotopic** if for all $i = 0, \dots, n$ there are maps $h_i : X_n \rightarrow Y_{n+1}$ such that $d_0 h_0 = f$ and $d_{n+1} h_n = g$ with the conditions:

$$d_i h_j = \begin{cases} h_{j-1} d_i & \text{if } i < j \\ d_i h_{i-1} & \text{if } i = j \neq 0, \\ h_j d_{i-1} & \text{if } i > j + 1 \end{cases}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j \\ h_j s_{i-1} & \text{if } i > j \end{cases}.$$

Remark 3.1. By some category theory (see remarks at 2.5) a simplicial set X can be thought of as made up of Δ^n . This means that to understand the mapping $X \times \Delta^1 \rightarrow Y$, one needs to understand the prism $\Delta^n \times \Delta^1$. And for the latter, one needs to understand how it is triangulated into $(n+1)$ -dimensional simplices. It is a standard fact that $\Delta^n \times \Delta^1$ is triangulated by exactly $n+1$ number of $(n+1)$ -simplices (see [Fri23], Products). Thus, h_j are just the maps defined on the $n+1$ distinct simplices that make up the triangulated prism.

Theorem 3.5. Let X, Y be two simplicial sets. Simplicial homotopy is an equivalence relation on maps $X \rightarrow Y$ whenever Y is a Kan complex. It is additive when Y simplicial abelian group.

Proof. The proof of this requires some combinatorial work, including introducing *shuffles*. For the proof, see the discussion preceding Corollary 6.11 in [May67], □

4 On $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$

In this section, we introduce the category on the right side of the Dold-Kan. We define chain complexes, chain maps; define what it means to be chain homotopic, define homology of a chain complex. This subsection is mostly for bookkeeping and as such, the facts provided are given concisely and without proof.

Definition 4.1. A **chain complex** (C_*, ∂) of R -modules is a family of R -modules $C_* =$

$\{C_n\}_{n \in \mathbb{Z}}$ with R -module homomorphisms $\partial_n : C_n \rightarrow C_{n-1}$ of order two, i.e. $\partial^2 \equiv 0$ where the map ∂ is called the **differential** of the chain complex.

A remark on notation is that for the rest of the project the sub-asterisk of C_* and the mention of differential is omitted. So a chain complex is denoted with C with the differential either one of $\partial, \partial', \tilde{\partial}, \bar{\partial}$; and $\partial^{\mathcal{F}}$ for differentials specific to some complex created under a functor $\mathcal{F}$.

Remark 4.1. An abelian group is exactly a $\mathbb{Z}$ -module. So a chain complex of abelian groups has $\mathbb{Z}$ -linear group homomorphisms as differentials.

Since this project is mostly concerned with the category of abelian groups, the definitions that follow hereafter are also adapted according to this. So, when a *chain complex* is given; unless explicitly stated, it is a chain complex of abelian groups.

Definition 4.2. For a chain complex C of abelian groups, define the group of n -cycles and the group of n -boundaries of C as $Z_n = Z_n(C) := \ker \partial_n$ and $B_n = B_n(C) := \text{im } \partial_{n+1}$, respectively. One has⁴ $0 \subseteq B_n \subseteq Z_n \subseteq C_n$ for all n . Then the **n th homology group** $H_n(C)$ of C is defined to be $H_n(C) := Z_n/B_n$.

Definition 4.3. A **chain map** $f : C \rightarrow D$ between chain complexes C and D is a family of abelian group homomorphisms $\{f_n : C_n \rightarrow D_n\}_{n \in \mathbb{Z}}$ such that it commutes with the differentials, i.e. $f \circ \partial = \partial \circ f$ whenever appropriate.

Definition 4.4. A chain complex C is called **non-negatively graded** if it is bounded below at $n = 0$. In other words, $C_n = 0$ for all $n < 0$.

Proposition 4.1. *The collection of chain complexes of abelian groups and chain maps between them forms a category, denoted $\mathbf{Ch}(\mathbf{Ab})$. The non-negatively graded chain complexes of abelian groups, denoted $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ is a subcategory.*

Isomorphism of chain complexes and the notion of a subcomplex are defined naturally.

Definition 4.5. Let C, C' be two chain complexes in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$. They are **isomorphic** if there exists a chain map $f : C \rightarrow C'$ such that $f_n : C_n \rightarrow C'_n$ is a group isomorphism for each $n \geq 0$.

⁴The analogue as R -submodules is true because $\partial^2 \equiv 0$.

Definition 4.6. Given chain complexes (C, ∂) and (C', ∂') in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$, C' is called a **subcomplex** of C if $C'_n \leq C_n$ for all $n \geq 0$ and $\partial' = \partial|_{C'}$.

Definition 4.7. Let (C, ∂) be a complex and a subcomplex pair with the differential of C given as ∂ . The **quotient complex** $\left(\left(C/C'\right)_*, \tilde{\partial}\right)$ is composed of $\left(C/C'\right)_n := C_n/C'_n$ and differential $\tilde{\partial}_n(c + C'_n) := \partial_n(c) + C'_{n-1}$ with $c \in C$.

Definition 4.8. Let (C, ∂) and (C', ∂') be two chain complexes. The **direct sum** $C \oplus C'$ of chain complexes consists of $(C \oplus D)_n := C_n \oplus D_n$ and differential $(\partial, \partial')_n := (\partial_n, \partial'_n)$.

Proposition 4.2. *If a chain complex C decompose into two subcomplexes N, D as isomorphism of abelian groups at each level n , then this decomposition is an isomorphism of chain complexes.*

Proof. Writing out the isomorphism given $\varphi(x+y) = x+y$ on $N_n \oplus D_n$ for each n , one just needs to check that φ is a chain map. This follows from N and D being subcomplexes. $\square$

Definition 4.9. Let C, D be chain complexes. A chain map $f : C \rightarrow D$ is called **nullhomotopic** if there exist maps $S_n : C_n \rightarrow D_{n+1}$ such that $f = \partial S + S\partial$. The maps $\{S_n\}$ are called **chain contractions** of f . Given another map $g : C \rightarrow D$, f and g are called **chain homotopic** if their difference $f - g$ is nullhomotopic, i.e. if $f - g = \partial S + S\partial$.

Definition 4.10. Given chain complexes C and D , a chain map $f : C \rightarrow D$ is called a **chain homotopy equivalence** if there exists a chain map $k : D \rightarrow C$ such that fk and kf are respectively chain homotopic to the identity chain maps of D and C .

5 Dold-Kan

Theorem 5.1. Let $\mathbf{Ab}$ be the category of abelian groups. Then the normalized chain complex functor $\mathcal{N}$ is an equivalence of categories between $s\mathbf{Ab}$ and $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$:

$$s\mathbf{Ab} = \left\{ \begin{array}{c} \text{simplicial} \\ \text{abelian groups} \end{array} \right\} \xrightarrow{\mathcal{N}} \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) = \left\{ \begin{array}{c} \text{chain} \\ \text{complexes in } \mathbf{Ab} \end{array} \right\}$$

Additionally, under $\mathcal{N}$:

- (1) Simplicial homotopy $\leftrightarrow$ homology, i.e. $\pi_*(A) \cong H_*(NA)$,
- (2) Simplicially homotopic maps $\leftrightarrow$ chain homotopic maps.

The proof of Dold-Kan is adapted and reconstructed from [GJ09] and [Wei94] while utilizing the MSc thesis of [Hug22]⁵.

5.1 Proof of Category Equivalence

To be able to establish category equivalence (see Definition at A.1), we need to construct the normalized chain complex functor $\mathcal{N} : s\mathbf{Ab} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ and the inverse functor $\mathcal{K} : \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) \rightarrow s\mathbf{Ab}$, and show that their compositions are naturally isomorphic to identity.

The normalized chain complex functor $\mathcal{N}$ is naturally isomorphic to the quotient $\mathcal{C}/\mathcal{D}$ of two other functors $\mathcal{C}, \mathcal{D} : s\mathbf{Ab} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ which are now introduced. The quotient description is explicitly used when proving one way of natural isomorphisms. For space constraints and as almost exactly the same things are done when proving well-definition of functors $\mathcal{C}, \mathcal{D}$, and $\mathcal{N}$; and as $\mathcal{N}$ is thematically more important, the proofs for $\mathcal{C}$ and $\mathcal{D}$ are shelved into Appendix A.2.

For a simplicial object, $\mathcal{C}(A)$ is also called the **Moore complex** of A .

Lemma 5.2. *Let A be a simplicial abelian group. Its **unnormalized chain complex** $\mathcal{C}(A)$ is defined to be, for each $n \geq 0$, $\mathcal{C}_n(A) = \mathcal{C}_n := A_n$ with differential of the chain complex $\partial^{\mathcal{C}} : \mathcal{C}_n \rightarrow \mathcal{C}_{n-1}$ given by $\partial^{\mathcal{C}} = \sum_{i=0}^n (-1)^i d_i$. The assignment $A \mapsto \mathcal{C}(A)$ and*

⁵For the latter, much care has been threaded not to plagiarize it.

$(f : A \rightarrow B) \mapsto f$ defines the unnormalized chain complex functor:

$$\mathcal{C} : s\mathbf{Ab} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

Proof. See Proof A.2. □

We also have the following.

Lemma 5.3. *Let A be a simplicial abelian group. Its **degenerate chain complex** $\mathcal{D}(A)$ is defined to be, for each $n \geq 0$, $\mathcal{D}_n(A) = \mathcal{D}_n := \sum_{i=0}^{n-1} s_i(C_{n-1}(A))$ with differential of the chain complex as $\partial^{\mathcal{D}} : \mathcal{D}_n \rightarrow \mathcal{D}_{n-1}$ given by $\partial^{\mathcal{D}} = \partial^{\mathcal{C}}|_{\mathcal{D}}$. The assignment $A \mapsto \mathcal{D}(A)$ and $(f : A \rightarrow B) \mapsto f|_{\mathcal{D}(A)}$ defines the degenerate chain complex functor:*

$$\mathcal{D} : s\mathbf{Ab} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

Additionally, $\mathcal{D}(A)$ is a subcomplex of $\mathcal{C}(A)$.

Proof. See Proof A.2 for well-definition. To see $\mathcal{D}(A)$ being a subcomplex of $\mathcal{C}(A)$, observe that for each n , $\mathcal{D}_n(A)$ is a subgroup of $C_n(A)$ by $\mathbb{Z}$ -linearity. Also it has the identity 0 by $s_i(0) = 0$. The differential condition $\partial^{\mathcal{D}} = \partial^{\mathcal{C}}|_{\mathcal{D}}$ follows by definition. Thus, it is a subcomplex by Definition 4.6. □

Here is the *normalized chain complex functor* $\mathcal{N}$ that we are interested in.

Proposition 5.4. *Let A be a simplicial abelian group. Its **normalized chain complex** $\mathcal{N}(A)$ is defined to be, for each $n \geq 0$, $\mathcal{N}_n(A) = \mathcal{N}_n := \bigcap_{i=0}^{n-1} \ker(d_i : A_n \rightarrow A_{n-1})$ with differential of the chain complex as $\partial^{\mathcal{N}} : \mathcal{N}_n \rightarrow \mathcal{N}_{n-1}$ given by $\partial^{\mathcal{N}} = (-1)^n d_n$. This defines the **normalized chain complex functor** $\mathcal{N}$, which is one of the two main functors in Dold-Kan:*

$$\mathcal{N} : s\mathbf{Ab} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

Additionally, $\mathcal{N}(A)$ is a subcomplex of $\mathcal{C}(A)$.

Proof. Let A be a simplicial abelian group. $\mathcal{N}_n(A)$ is a normal subgroup of A as $\cap$ of $\ker$ is a normal subgroup. In particular, $\mathcal{N}_n(A)$ is an abelian group.

∂ having the correct codomain amounts to the statement that for any $x \in \mathcal{N}_n(A)$,

one has $d_i(\partial(x)) = 0$ for all i with $0 \leq i \leq n-2$. Given such i :

$$\begin{aligned} d_i \partial(x) &= (-1)^n d_i d_n(x) \\ &= (-1)^n d_{n-1} d_i(x) && \text{by simplicial identities} \\ &= 0 \end{aligned}$$

because $x \in \mathcal{N}_n(A)$, so $d_i(x) = 0$ for any such i . ∂ being a group homomorphism again follows from $\mathbb{Z}$ -linearity.

To see ∂ being of order two:

$$\begin{aligned} \partial_n \circ \partial_{n+1}(\mathcal{N}_{n+1}(A)) &= (-1)^{2n+1} d_n d_{n+1}(\mathcal{N}_{n+1}(A)) && \text{by definition} \\ &= -d_n d_n(\mathcal{N}_{n+1}(A)) && \text{by simplicial identities} \\ &= -d_n(0) = 0 && \text{as } \mathcal{N}_{n+1} \text{ contains } \ker d_n \end{aligned}$$

Given a simplicial map $f : A \rightarrow B$, to see that given any $x \in \mathcal{N}_n(A)$, we have $f_n(x) \in \mathcal{N}_n(B)$, first observe that $d_i(x) = 0$ for any i with $0 \leq i \leq n-1$. And $d_i(f_n(x) = f_n(d_i(x))) = 0$, for the fact that simplicial maps commute with face maps, which gives us the fact that $f_n(x) \in \mathcal{N}_n(B)$.

The chain map condition amounts to showing that the following square commutes:

$$\begin{array}{ccccc} \dots & \longleftarrow & \mathcal{N}_{n-1}(A) & \xleftarrow{\partial=(-1)^n d_n} & \mathcal{N}_n(A) & \longleftarrow & \dots \\ & & \downarrow \mathcal{N}(f)_{n-1} & & \downarrow \mathcal{N}(f)_n & & \\ \dots & \longleftarrow & \mathcal{N}_{n-1}(B) & \xleftarrow{\partial=(-1)^n d_n} & \mathcal{N}_n(B) & \longleftarrow & \dots \end{array}$$

which is due to simplicial maps commuting with face maps, hence the restricted map $\mathcal{N}(f) = f|_{\mathcal{N}(A)}$ commutes with the differential ∂ .

For functoriality of $\mathcal{N}$, composition and identity laws must be checked.

The identity morphism $\mathbb{1}_A$ on a simplicial set A is a natural transformation, hence its components at each $n \geq 0$ are identity functions $\mathbb{1}_{A_n}$. Then, it suffices to observe, by definition of $\mathcal{N}$:

$$\mathcal{N}(\mathbb{1}_A)_n = \mathbb{1}_{A_n}|_{\mathcal{N}_n} = \mathbb{1}_{\mathcal{N}(A)_n}$$

for each $n \geq 0$. So $\mathcal{N}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{N}(A)}$.

Consider simplicial abelian groups A, B, C and simplicial maps $f : A \rightarrow B$ and

$g : B \rightarrow C$. Then, given $x \in \mathcal{N}_n(A)$, observe that:

$$\begin{aligned}
\mathcal{N}(g \circ f)_n(x) &= (g \circ f)|_{\mathcal{N}(A)_n}(x) \\
&= (g \circ f)(x) && \text{by choice of } x \\
&= g(f(x)) && \text{evaluating at } f(x) \\
&= g(f|_{\mathcal{N}(A)_n}(x)) && \text{again, by choice of } x \\
&= g|_{\mathcal{N}(B)_n}(f|_{\mathcal{N}(A)_n}(x)) && \text{by } f|_{\mathcal{N}(A)_n}(x) \in \mathcal{N}(B)_n \\
&= (g|_{\mathcal{N}(B)_n} \circ f|_{\mathcal{N}(A)_n})(x) && \text{by definition of composition} \\
&= (\mathcal{N}(g) \circ \mathcal{N}(f))(x).
\end{aligned}$$

Hence, $\mathcal{N}$ is functorial.

To show $\mathcal{N}(A)$ being a subcomplex of $C(A)$, it is only needed to show that $\partial^{\mathcal{N}} = \partial^C|_{\mathcal{N}A}$ for any $A \in s\mathbf{Ab}$ by definition of a subcomplex (4.6). This directly follows from definitions and simplicial identities. $\square$

The following relates $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{N}$.

Lemma 5.5. *Let A be a simplicial abelian group. Then we have $\mathcal{C}(A) = \mathcal{N}(A) \oplus \mathcal{D}(A)$, hence:*

$$\mathcal{N}(A) \cong \mathcal{C}(A) / \mathcal{D}(A)$$

Proof. We need to show that, for any $n \geq 0$, the intersection $\mathcal{N}_n \cap \mathcal{D}_n$ is zero and $\mathcal{C}_n$ can be decomposed into a normal and a degenerate part.

Let $y \in \mathcal{N}_n \cap \mathcal{D}_n$ be a nonzero element. Since $y \in \mathcal{D}_n$, $y = \sum_{i=0}^{n-1} s_i x_i$ for elements $x_i \in \mathcal{C}_{n-1}$. As $y \neq 0$, we can choose an index k for $0 \leq k \leq n-1$ to be the smallest integer such that $s_k x_k \neq 0$. Observe the following:

$$\begin{aligned}
d_k y &= d_k \left(s_k x_k + \sum_{i=k+1}^{n-1} s_i x_i \right) \\
\implies 0 &= d_k s_k x_k + \sum_{i=k+1}^{n-1} d_k s_i x_i && \text{by } y \in \mathcal{N}_n \text{ and } \mathbb{Z}\text{-linearity} \\
\implies 0 &= x_k + \sum_{i=k+1}^{n-1} s_{i-1} d_k x_i && \text{by simplicial identity for } d_k s_k \\
\iff x_k &= - \sum_{i=k+1}^{n-1} s_{i-1} d_k x_i
\end{aligned}$$

Applying s_k :

$$\begin{aligned}
s_k x_k &= - \sum_{i=k+1}^{n-1} s_k s_{i-1} d_k x_i && \text{by simplicial identity for } s_k s_{i-1} \\
&= - \sum_{i=k+1}^{n-1} s_i s_k d_k x_i \\
&= - \sum_{i=k+1}^{n-1} s_i x_i \neq 0 && \text{by simplicial identity for } s_k d_k \text{ \& choice of } k
\end{aligned}$$

However, when the obtained expression for $s_k x_k$ is substituted into $y \in \mathcal{D}_n$, y can be written completely in terms of $(k+1)$ + degeneracies. This contradicts the choice of k as the smallest such index. Therefore $\mathcal{N}_n \cap \mathcal{D}_n = 0$.

Let j in $0 \leq j \leq n$ be the smallest integer such that $d_j y \neq 0$, if it does not exist for y it means y is completely in $\mathcal{N}_n$ and we are done. The proof for the decomposition $\mathcal{N}_n + \mathcal{D}_n = \mathcal{C}_n$ is done inductively on the integer j . Consider the element y' :

$$y' := y - s_j d_j y$$

The claim is that y' is congruent to an element of $\mathcal{N}_n$ so that with the obvious $s_j d_j y \in \mathcal{D}_n$, we can decompose y . Fix j as such. Then for $i < j$:

$$\begin{aligned}
d_i y' &= d_i - d_i s_j d_j y \\
&= d_i y - s_{j-1} d_i d_j y && \text{by simplicial identity for } d_i s_j \\
&= d_i y - s_{j-1} d_{j-1} d_i y && \text{by simplicial identity for } d_i d_j \\
&= 0 && \text{by choice of } j
\end{aligned}$$

And for $i = j$, $d_j y' = 0$ by simplicial identities again. Pushing j to n in the same fashion, y' is congruent to an element of $\mathcal{N}_n$ modulo $\mathcal{D}_n$.

To conclude chain complex isomorphism $\mathcal{C} = \mathcal{N} \oplus \mathcal{D}$, observe that we have canonical group homomorphisms $\mathcal{C}_n \rightarrow \mathcal{N}_n \oplus \mathcal{D}_n$ and their inverses. And as $\mathcal{N}, \mathcal{D}$ being subcomplexes of $\mathcal{C}$ through Lemma 5.3 and Proposition 5.4; by Proposition 4.2, this decomposition is a chain complex isomorphism. The quotient expression follows by definition of a quotient complex (4.7). $\square$

We also include a sketch of how $\mathcal{N}$ is obtained consecutively by applying $\mathcal{C}$ and $m/\mathcal{D}$.

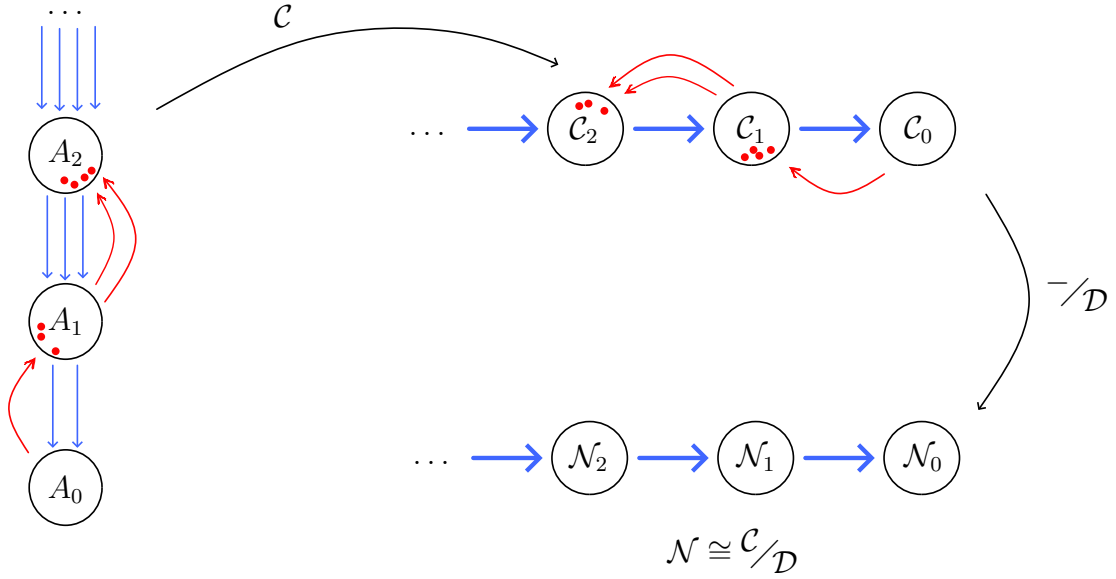


Figure 7: Vertical structure is the simplicial set where blue thin arrows are face maps; reds are degeneracy maps; and thick blue arrow is the differential obtained via $\mathcal{C}$. Red dots are degenerate simplices.

Now the inverse functor $\mathcal{K}$ is constructed, which needs to be a functor that sends chain complexes to simplicial abelian groups and chain maps to simplicial maps.

First, the definition of $\mathcal{K}$ is introduced on objects. Then, the definition is elucidated by expanding the definition on lower dimensions. After, it is explained how the object assignment naturally gives rise to the morphism assignments of $\mathcal{K}$. Lastly, functoriality of $\mathcal{K}$ is proven.

Definition 5.1 (The recipe for $\mathcal{K}$; [Wei94]). Let a be chain complex $C \in \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$. For each $n \geq 0$, $\mathcal{K}_n(C)$ is defined to be:

$$\mathcal{K}_n(C) := \bigoplus_{p \leq n} \bigoplus_{\eta} C_p^{\eta}$$

where η is any monotonic surjection $\eta : [n] \twoheadrightarrow [p]$ and C_p^{η} is a copy of the abelian group C_p indexed by η .

For some $\alpha : [m] \rightarrow [n]$ in Δ , $\mathcal{K}(C)(\alpha) : \mathcal{K}_n(C) \rightarrow \mathcal{K}_m(C)$ is defined on its restrictions to $\mathcal{K}(C)(\alpha)|_{\eta} : C_p^{\eta} \rightarrow \mathcal{K}_m(C)$ for each surjection $\eta : [n] \twoheadrightarrow [p]$, via the epi-monic

factorization of $\eta\alpha = \epsilon\eta'$:

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \eta' \downarrow & & \downarrow \eta \\ [q] & \xrightarrow{\epsilon} & [p] \end{array}$$

The definition of $\mathcal{K}(C)(\alpha)|_\eta$ is given as:

$$\mathcal{K}(C)(\alpha)|_\eta \equiv \begin{cases} \varphi : C_p^\eta \xrightarrow{\sim} C_p^{\eta'} \subseteq \mathcal{K}_m(C), & \text{if } p = q; \\ \partial_p : C_p \rightarrow C_{p-1} \cong C_q^{\eta'}, & \text{if } p = q + 1 \text{ and } \epsilon = \epsilon_p; \\ 0, & \text{otherwise.} \end{cases}$$

Here φ is the natural identification of C_p^η with $C_p^{\eta'}$, ∂_p is the differential of the chain complex C , and ϵ_p denotes the injection $[p] \hookrightarrow [p+1]$ that omits the last vertex of $[p+1]$ (or the p th coface map). The maps $\mathcal{K}(C)(\alpha)$ are called **simplicial structure maps** of $\mathcal{K}(C)$ ([GJ09]).

Before proving that the given recipe forms a well-defined simplicial abelian group and $\mathcal{K}$ a well-defined functor $\mathbf{Ch}_{\geq 0}(\mathbf{Ab}) \rightarrow s\mathbf{Ab}$, it needs to be elucidated.

Example 5.1. Take a chain complex $C \in \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$. Observe that for $n = 0$, there exists a unique surjection, only on $p = 0$:

$$\begin{array}{ccc} \eta : [0] & \longrightarrow & [0] \\ 0 & \longmapsto & 0 \end{array}$$

which results in $\mathcal{K}_0 = C_0^\eta$.

For $n = 1$, there exist two surjections, one for $p = 0$, one for $p = 1$:

$$\begin{array}{ccc} \eta : [1] \rightarrow [0] & & \eta : [1] \rightarrow [1] \\ 0 \longmapsto 0 & & 0 \longmapsto 0 \\ 1 \longmapsto 0 & & 1 \longmapsto 1 \end{array}$$

which results in $\mathcal{K}_1 = C_0 \oplus C_1$.

For $n = 2$, $\mathcal{K}_2(C)$ is sum of four abelian groups, through:

$$\begin{array}{cccc}
 \eta_a : [2] \longrightarrow [0] & \eta_b : [2] \longrightarrow [1] & \eta_c : [2] \longrightarrow [1] & \eta_d : [2] \longrightarrow [2] \\
 \begin{array}{c} 0 \longrightarrow 0 \\ 1 \searrow \nearrow \\ 2 \searrow \nearrow \end{array} & \begin{array}{c} 0 \longrightarrow 0 \\ 1 \searrow \nearrow \\ 2 \searrow \nearrow \end{array} & \begin{array}{c} 0 \longrightarrow 0 \\ 1 \longrightarrow 1 \\ 2 \searrow \nearrow \end{array} & \begin{array}{c} 0 \longrightarrow 0 \\ 1 \longrightarrow 1 \\ 2 \longrightarrow 2 \end{array}
 \end{array}$$

which results in $\mathcal{K}_2(C) = C_0 \oplus (C_1)^2 \oplus C_2$.

It is now apparent that for $\mathcal{K}_n(C)$ we get a sum of 2^n abelian groups with number of p -indexed groups as binomial coefficients $\binom{n}{p}$. This gives us the (Pascal) triangle of $\mathcal{K}_n(C)$:

$$\begin{array}{ccccccc}
 \mathcal{K}_0(C) & \longrightarrow & = & \longrightarrow & C_0 & & \\
 \mathcal{K}_1(C) & \longrightarrow & = & \longrightarrow & C_0 & \oplus & C_1 \\
 \mathcal{K}_2(C) & \longrightarrow & = & \longrightarrow & C_0 & \oplus & (C_1)^2 \oplus C_2 \\
 \mathcal{K}_3(C) & = & C_0 & \oplus & (C_1)^3 & \oplus & (C_2)^3 \oplus C_3 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots
 \end{array}$$

The structure so far is obviously not sufficient to make up a simplicial abelian group. Recall that a simplicial abelian group A is a contravariant functor $\Delta \rightarrow \mathbf{Ab}$ which assigns a group homomorphism $A(\alpha) : A_n \rightarrow A_m$ for each nondecreasing monotonic map $\alpha : [m] \rightarrow [n]$ in Δ (c.f. Definitions 2.8 and 2.11). This is the approach from [Wei94] for constructing structure maps $\mathcal{K}(C)(\alpha)$, given in Definition 5.1.

The present goal is to clarify how the maps $\mathcal{K}(C)(\alpha)$ are defined. Observe the following lemma.

Lemma 5.6. *Given a map $\alpha : [m] \rightarrow [n]$ in Δ , since $\mathcal{K}_n(C)$ is a direct sum of abelian groups for each $n \geq 0$, it is sufficient to define maps $\mathcal{K}(C)(\alpha) : \mathcal{K}_n(C) \rightarrow \mathcal{K}_m(C)$ restricted to each abelian group C_p^η as $\mathcal{K}(C)(\alpha)|_\eta$.*

Proof. It follows by the universal property of direct sum of $\mathbb{Z}$ -modules. In other words, by definition of direct sum, an element of $\mathcal{K}_n(C)$ can be written uniquely as a sum of elements, each from a summand C_p^η . Thus, a collection of maps $\{f|_\eta\}_\eta$ on $\{C_p^\eta\}_\eta$, out of the sum, determines uniquely a map $\mathcal{K}(C)(\alpha)$ on $\mathcal{K}_n(C)$. $\square$

Example 5.2. The definition given in 5.1 for $\mathcal{K}(C)(\alpha)|_\eta$ is demonstrated for $m = 4$ and $n = 2$. Consider the nondecreasing monotonic map $\alpha : [4] \rightarrow [2]$ given by:

$$\alpha : [4] \longrightarrow [2]$$

$$0, 1 \longmapsto 0$$

$$2, 3, 4 \longmapsto 1$$

To determine the ultimate map, the epi-monic factorization $\epsilon\eta'$ of $\eta\alpha$ needs to be found for each η as given in the Example 5.1 for $\mathcal{K}_2(C)$. Considering the four possible surjections η , one obtains the following four factorizations.

$$\begin{array}{ccc} \begin{array}{ccc} [4] & \xrightarrow{\alpha} & [2] \\ \eta' \downarrow & & \downarrow \eta_a \\ [0] & \xrightarrow{\text{id}} & [0] \end{array} & \begin{array}{ccc} [4] & \xrightarrow{\alpha} & [2] \\ \eta' \downarrow & & \downarrow \eta_b \\ [0] & \xrightarrow{\epsilon_0} & [1] \end{array} & \begin{array}{ccc} [4] & \xrightarrow{\alpha} & [2] \\ \eta' \downarrow & & \downarrow \eta_c \\ [1] & \xrightarrow{\text{id}} & [1] \end{array} & \begin{array}{ccc} [4] & \xrightarrow{\alpha} & [2] \\ \eta' \downarrow & & \downarrow \eta_d \\ [1] & \xrightarrow{\epsilon_1} & [2] \end{array} \end{array}$$

The map $\mathcal{K}(C)(\alpha) : \mathcal{K}_2(C) \rightarrow \mathcal{K}_4(C)$ is defined according to these factorizations on each C_p^η . It can be observed from the cases of Definition 5.1 and the factorizations above that we get the following. The subscripts are omitted from η for clarity.

$$\mathcal{K}(C)(\alpha)|_{\eta_i} \equiv \begin{cases} \varphi : C_0^\eta \xrightarrow{\sim} C_0^{\eta'}, & \text{if } i = a; \\ \partial_1 : C_1 \rightarrow C_0, & \text{if } i = b; \\ \varphi' : C_1^\eta \xrightarrow{\sim} C_1^{\eta'}, & \text{if } i = c; \\ \partial_2 : C_2 \rightarrow C_1, & \text{if } i = d. \end{cases}$$

Omitting the indexing by η altogether, this results in the following mapping:

$$\begin{array}{ccccccc} \mathcal{K}_2(C) & = & C_0 & \oplus & C_1 & \oplus & C_1 & \oplus & C_2 \\ \downarrow \mathcal{K}(C)(\alpha) & & \downarrow \simeq & \swarrow \partial_1 & \downarrow \simeq & \swarrow \partial_2 & \downarrow \simeq & & \\ \mathcal{K}_4(C) & = & C_0 & \oplus & C_1 & \oplus & (C_1)^3 & \oplus & (C_2)^6 & \oplus & (C_3)^4 & \oplus & C_4 \end{array}$$

The reason for why φ with ∂_1 and φ' with ∂_2 map together is because the surjections η' determining them are identical (*c.f.* the factorization squares for $\eta\alpha$).

Proposition 5.7. *Given a chain complex $C \in \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$, $\mathcal{K}(C)$ is a simplicial abelian group as described in Definition 5.1 and as elucidated in Example 5.2.*

Proof. First, $\mathcal{K}(C)$ is composed of abelian groups $\mathcal{K}_n(C)$ given their construction and maps $\mathcal{K}(C)(\alpha)$ are group homomorphisms as $(\varphi, \partial, 0)$ all also are. From these and by Lemma 5.6, there is a well-defined structure.

Next, showing that $\mathcal{K}(C)$ is a simplicial abelian group amounts to showing functoriality for $\Delta^{op} \rightarrow \mathbf{Ab}$. Observe that for any surjection $\eta : [n] \rightarrow [p]$ we have the epi-monic factorization $\eta \mathbb{1}_n = \mathbb{1}_p \eta$. Due to $p = q$, we get the identity map on C_p^η $\varphi = \text{id} : C_p^\eta \rightarrow C_p^\eta$, which makes up the identity map on whole of $\mathcal{K}_n(C)$. Thus, $\mathcal{K}(C)(\mathbb{1}_{[n]}) = \mathbb{1}_{\mathcal{K}_n(C)}$ and the identity law is shown.

For the composition law, consider $\alpha : [m] \rightarrow [n]$ and $\beta : [n] \rightarrow [l]$. The goal is to show that $\mathcal{K}(C)(\beta\alpha) = \mathcal{K}(C)(\alpha)\mathcal{K}(C)(\beta)$. Observe that it is sufficient to show this by restricting to one arbitrary surjection $\eta : [l] \rightarrow [p]$. The proof is straightforward definition-checking: One can find the unique epi-monic factorization of $\eta(\beta\alpha)$ and other necessary compositions such as of $\eta\beta$ and $\eta'\alpha$ (where η' is created by factorizing $\eta\beta$). Since factorization squares can be composed together, the composition is checked from Definition 5.1 and it follows that $\mathcal{K}(C)$ obeys the composition law from proof by cases. $\square$

Proposition 5.8. *The construction $\mathcal{K} : \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) \rightarrow s\mathbf{Ab}$ is a functor. On objects, it assigns the simplicial abelian group $\mathcal{K}(C)$ to a non-negatively graded chain complex C with simplicial structure maps $\mathcal{K}(C)(-)$ defined as in Definition 5.1. On morphisms, it assigns a chain map $f : C \rightarrow D$ to the induced simplicial map $\mathcal{K}(f) : \mathcal{K}(C) \rightarrow \mathcal{K}(D)$, which is defined component-wise.*

Proof. For well-definition of $\mathcal{K}$ see that by Proposition 5.7, for any $C \in \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$, $\mathcal{K}(C)$ is a simplicial abelian group.

Given a chain map $f : C \rightarrow C'$, $\mathcal{K}(f)$ is defined component-wise, i.e. $f_p : C_p \rightarrow C'_p$ is defined on each corresponding summand C_p^η . Since a chain map f commutes with the differential ∂ , the identity and the zero map; which constitute $\mathcal{K}(f)$, f commutes with the structure maps. Hence $\mathcal{K}(f)$ is a natural transformation of functors $\mathcal{K}(C)$ and $\mathcal{K}(C')$,

i.e. the following square commutes:

$$\begin{array}{ccc} \mathcal{K}_n(C) & \xrightarrow{\mathcal{K}_n(f)} & \mathcal{K}_n(C') \\ \mathcal{K}(C)(\alpha) \downarrow & & \downarrow \mathcal{K}(C')(\alpha) \\ \mathcal{K}_m(C) & \xrightarrow{\mathcal{K}_m(f)} & \mathcal{K}_m(C') \end{array}$$

for any morphism $\alpha : [m] \rightarrow [n]$ in Δ .

The identity and composition laws for functoriality follow immediately. This is because the simplicial maps $\mathcal{K}(f)$ are defined component-wise and they inherit the component-level identity and composition laws of the underlying category of non-negatively graded chain complexes. $\square$

One can obtain the face and degeneracy maps of $\mathcal{K}(C)$ for a given non-negatively graded chain complex C just by definition of structure maps $\mathcal{K}(C)(-)$.

Remark 5.1. By the definition of a simplicial object as a contravariant functor out of Δ , the face maps d_i and degeneracy maps s_j of $\mathcal{K}(C)$ are given by $d_i = \mathcal{K}(C)(d^i) = \mathcal{K}(C)(\epsilon_i)$ and $s_j = \mathcal{K}(C)(s^j) = \mathcal{K}(C)(\eta_j)$, i.e. as images of the coface and codegeneracy maps in Δ . The latter is in the notational convention of [Wei94] and the former is in [GJ09], [Fri23], [May67] (also *c.f.* Remark 2.3 and Example 2.5).

To show category equivalence, the following natural isomorphisms need to be proven:

$$\mathcal{N}\mathcal{K} \cong \mathbb{1}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab})} \qquad \mathcal{K}\mathcal{N} \cong \mathbb{1}_{s\mathbf{Ab}}$$

Proposition 5.9. *Let $C \in \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$. Then $\mathcal{N}\mathcal{K}(C) \cong C$ as chain complexes.*

Proof. Fix some $n \geq 0$ and let $C \in \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$. Recall that $\mathcal{K}_n(C) = \bigoplus_{p \leq n} \bigoplus_{\eta} C_p^{\eta}$ for surjections $\eta : [n] \twoheadrightarrow [p]$.

To establish the isomorphism, first, the level-wise isomorphism between groups $\mathcal{N}_n(\mathcal{K}(C))$ and $\mathcal{C}_n$ is shown. Second, it is shown that this level-wise isomorphism preserves the complex structure, by proving that the differential of $\mathcal{N}\mathcal{K}(C)$ coincides with the differential of $\mathcal{C}$.

Recall the definition of $\mathcal{N}(A)$. Its levels are defined to be:

$$\mathcal{N}(A)_n = \mathcal{N}_n := \bigcap_{i=0}^{n-1} \ker(d_i : A_n \rightarrow A_{n-1})$$

By this, understanding $\mathcal{NK}(C)$ amounts to understanding the intersection of the kernels of the face maps $d_i : \mathcal{K}_n(C) \rightarrow \mathcal{K}_{n-1}(C)$, which is the same as trying to understand an arbitrarily large number of face maps on the groups C_p^η . However, this complexity can be reduced by directly examining the groups C_p^η and observe what happens when $p < n$ and $p = n$. In other words, we have the following two cases:

$$\mathcal{K}_n(C) = \left(\bigoplus_{p < n} C_p^\eta \right) \oplus C_n^{1[n]} \quad (3)$$

Let p be such that $0 \leq p < n$. Since the epi-monic factorization of $\mathbb{1}_{[p]}\eta$ is itself, the monic part ϵ_q is $\mathbb{1}_{[p]}$. So by definition of structure maps $\mathcal{K}(C)(\alpha)$, this gives $\mathcal{K}(C)(\eta)|_{\mathbb{1}_p}$ to be the natural association $C_p^{\mathbb{1}_{[p]}} \rightarrow C_p^\eta$. Equivalently,

$$C_p^\eta = \mathcal{K}(C)(\eta) \left(C_p^{\mathbb{1}_{[p]}} \right)$$

Additionally, observe that since $p < n$ strictly, $\eta = H$ where H is a composition of some codegeneracy maps. Under $\mathcal{K}$, H is mapped contravariantly into a composition of degeneracy maps of $\mathcal{K}(C)$, denote this by S , (e.g. $S = s_1 s_2 s_5$). Combining these observations together:

$$C_p^\eta = S C_p^{\mathbb{1}_{[p]}}$$

But then, $C_p^\eta \subseteq D_n(\mathcal{K}(C))$. And by $\mathcal{N}_n \oplus \mathcal{D}_n = \mathcal{C}_n$ (see Lemma 5.5), we have:

$$C_p^\eta \cap \mathcal{N}_n(\mathcal{K}(C)) = 0 \quad (4)$$

Hence, the left term on Eq. (3) is eliminated.

Let $p = n$. In this case, the face maps can be directly evaluated. Observe that $d_i = \mathcal{K}(C)(d^i)$ given a coface map $d^i : [n-1] \hookrightarrow [n]$ and $d_i : \mathcal{K}_n(C) \rightarrow \mathcal{K}_{n-1}(C)$ is the zero map unless $i = n$ by definition of $\mathcal{K}(C)(\alpha)$ (5.1). Particularly, the elements in $C_n^{1[n]}$ are mapped to 0 under d_i for all $i \neq n$. This means $C_n^{1[n]} \subseteq \mathcal{N}_n(\mathcal{K}(C))$. And since we also only have $C_n^{1[n]}$ in $\mathcal{K}_n(C)$ from Eq. (4), we have equality $C_n^{1[n]} = \mathcal{N}_n(\mathcal{K}(C))$.

$C_n^{1[n]} = \mathcal{N}_n(\mathcal{K}(C))$ means that $\mathcal{N}_n(\mathcal{K}(C))$ is composed of abelian groups. Recall, inside a simplicial abelian group $\mathcal{N}(A)$, the differential $\tilde{\partial}$ is given by $\tilde{\partial} = \sum_{i=0}^n (-1)^i d_i$ where d_i

are face maps of A . In the case of $\mathcal{N}(\mathcal{K}(C))$, this turns into:

$$\tilde{\partial} : \mathcal{N}_n(\mathcal{K}(C)) \rightarrow \mathcal{N}_{n-1}(\mathcal{K}(C))$$

given by the same sum where $d_i : \mathcal{K}_n(C) \rightarrow \mathcal{K}_{n-1} \rightarrow C$. Since from previous observation $d_i \equiv 0$ for all i with $i \neq n$, the differential reduces to:

$$\tilde{\partial} = (-1)^n d_n = (-1)^n \mathcal{K}(C)(d^n) \quad (5)$$

where the latter equality is from the definition of face maps of $\mathcal{K}(C)$ from Remark 5.1. As shown that $\mathcal{N}_n(\mathcal{K}(C)) = C_n^{1[n]}$, it suffices to evaluate $\mathcal{K}(C)(d^n)|_{1[n]}$ as demonstrated in Example 5.2. One looks at the epi-monic factorization of $d^n \mathbb{1}_{[n]}$, given with the following square:

$$\begin{array}{ccc} [n-1] & \xhookrightarrow{d^n} & [n] \\ \mathbb{1}_{[n-1]} \downarrow & & \downarrow \mathbb{1}_{[n]} \\ [n-1] & \xhookrightarrow{d^n} & [n] \end{array}$$

which, by Definition 5.1, gives:

$$\mathcal{K}(C)(d^n)|_{1[n]} \equiv \partial : C_n \rightarrow C_{n-1}$$

so that, $\tilde{\partial} = (-1)^n \partial$. It follows that $\tilde{\partial}^2 \equiv 0$. Therefore, $\mathcal{N}\mathcal{K}(C)$ is a non-negatively graded chain complex with differential $\tilde{\partial}$.

To establish the isomorphism $\mathcal{N}\mathcal{K}(C) \cong C$, one needs to work around the term $(-1)^n$ in $\tilde{\partial}$. This is solved via writing an explicit chain complex isomorphism that manages this. The following is directly taken from [Hug22]: Let $f_n : C_n \rightarrow \mathcal{N}_n(\mathcal{K}(C))$ be $f_n \equiv \mathbb{1}_{C_n}$ if $n \equiv 0$ or $3 \pmod{4}$; and $f_n \equiv -\mathbb{1}_{C_n}$ otherwise. This defines a well-defined chain complex isomorphism $f : C \rightarrow \mathcal{N}\mathcal{K}(C)$ with its inverse as itself. $\square$

Proposition 5.10. $\mathcal{N}\mathcal{K}$ is naturally isomorphic to $\mathbb{1}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab})}$

Proof. Denote the isomorphism constructed on objects in Proposition 5.9 with $\Psi : \mathcal{N}\mathcal{K} \rightarrow \mathbb{1}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab})}$. These are natural in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$. In other words, recall the morphism assignment of $\mathcal{K}$ to be component-wise on groups C_p^η . Given a chain map $f \in \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$, since

it suffices to look at components of the chain map $\mathcal{N}\mathcal{K}_n(f)$, one has the following:

$$\begin{aligned}
 \mathcal{N}\mathcal{K}_n(f) &= \mathcal{K}(f)|_{\mathcal{N}_n(\mathcal{K}(C))} && \text{By definition of } \mathcal{N} \\
 &= \mathcal{K}(f)|_{C_n} && \text{From Proposition 5.9} \\
 &= f_n && \text{By component-wise definition of } \mathcal{K}
 \end{aligned}$$

So that given a chain map $f : C \rightarrow D$, the following square commutes:

$$\begin{array}{ccc}
 \mathcal{N}\mathcal{K}(C) & \xrightarrow{\Psi} & \mathbb{1}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab})}(C) \\
 \mathcal{N}\mathcal{K}(f) \downarrow & & \downarrow \mathbb{1}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab})}(f) \\
 \mathcal{N}\mathcal{K}(D) & \xrightarrow{\Psi} & \mathbb{1}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab})}(D)
 \end{array}$$

Recall from Proposition 5.9 that Ψ_n acts as either $\mathbb{1}$ or $-\mathbb{1}$ depending on $n \pmod{4}$. Because f_n is $\mathbb{Z}$ -linear, it commutes with multiplication by ± 1 . Ultimately, by definition of a natural isomorphism (see Definition A.2), the natural isomorphism is established. $\square$

Now the second natural isomorphism $\mathcal{K}\mathcal{N} \cong \mathbb{1}_{s\mathbf{Ab}}$ is to be shown. One first needs a lemma.

Lemma 5.11. *Let $f : B \rightarrow A$ be a simplicial map between simplicial abelian groups A and B . If $\mathcal{N}f : \mathcal{N}(B) \rightarrow \mathcal{N}(A)$ is an isomorphism of chain complexes, then f is an isomorphism of simplicial abelian groups.*

Proof. The proof of this is quite involved and not explored in this project. The reader is referred to [Wei94], Lemma 8.4.5. $\square$

Proposition 5.12. *$\mathcal{K}\mathcal{N}$ is naturally isomorphic to $\mathbb{1}_{s\mathbf{Ab}}$.*

Proof. The statement of the proposition is equivalent to the statement that $\mathcal{K}\mathcal{N}(A)$ is naturally isomorphic to A for every simplicial abelian group A .

To this end, a natural morphism of simplicial abelian groups $\Psi_A : \mathcal{K}\mathcal{N}(A) \rightarrow A$ is constructed which is then shown to be an isomorphism by Lemma 5.11 for every $A \in s\mathbf{Ab}$.

Ψ_A is defined as group homomorphisms on levels $\mathcal{K}\mathcal{N}_n(A)$. The homomorphisms are defined component-wise as $\mathcal{K}\mathcal{N}_n(A)$ is composed of direct sums of groups $\mathcal{N}_p(A)^\eta$, copies of $\mathcal{N}_p(A)$ indexed by surjections $\eta : [n] \twoheadrightarrow [p]$. So to define a morphism of simplicial

abelian groups, it suffices to define:

$$\Psi_{n,A} : \mathcal{N}_p(A)^\eta \rightarrow A_n$$

This map is defined in two steps: first, considering the isomorphic prototype $\mathcal{N}_p(A)$ instead of $\mathcal{N}_p(A)^\eta$, one applies the natural inclusion $\mathcal{N}_p(A) \hookrightarrow A_p$, second, the contravariant image of $\eta : [n] \rightarrow [p]$ under A is applied. In other words $\Psi_{n,A}$ is the following:

$$\mathcal{N}_p(A)^\eta \cong \mathcal{N}_p(A) \hookrightarrow A_p \xrightarrow{A(\eta)} A_n \quad (6)$$

So for any $x \in \mathcal{N}_p(A)^\eta$, $\Psi_{n,A}(x) = A(\eta)(x)$. This map is well-defined on $\mathcal{KN}_n(A)$ by Lemma 5.6. To show that this collection defines a well-defined simplicial map Ψ_A , is the same as showing that given any morphism $\alpha : [m] \rightarrow [l]$, Ψ_A commutes with α . Because Ψ_A is a natural transformation of simplicial abelian groups, i.e. the following square should commute:

$$\begin{array}{ccc} \mathcal{KN}_l(A) & \xrightarrow{\Psi_{l,A}} & A_l \\ \mathcal{KN}(\alpha) \downarrow & & \downarrow A(\alpha) \\ \mathcal{KN}_m(A) & \xrightarrow{\Psi_{m,A}} & A_m \end{array}$$

Since the maps $\Psi_{n,A}$ are defined on summands $\mathcal{N}_p(A)^\eta$, one can expand on this diagram by finding the epi-monic factorization $\epsilon\eta'$ of $\eta\alpha$, to obtain the diagram:

$$\begin{array}{ccccccc} & & & \xrightarrow{\Psi_{n,A}|_\eta} & & & \\ \mathcal{KN}_l(A) & \hookrightarrow & \mathcal{N}_p(A)^\eta \cong \mathcal{N}_p(A) & \hookrightarrow & A_p & \xrightarrow{A(\eta)} & A_l \\ \mathcal{KN}(\alpha) \downarrow & & \textcolor{red}{\mathcal{KN}(\alpha)|_\eta} \downarrow & & A(\epsilon) \downarrow & & \downarrow A(\alpha) \\ \mathcal{KN}_m(A) & \hookrightarrow & \mathcal{N}_q(A)^{\eta'} \cong \mathcal{N}_q(A) & \hookrightarrow & A_q & \xrightarrow{A(\eta')} & A_m \\ & & & \xleftarrow{\Psi_{m,A}|_{\eta'}} & & & \end{array}$$

By the component-wise definition, it suffices to show the commutativity of the red square which is equivalent to showing that:

$$(\Psi_{m,A}|_{\eta'} \circ \mathcal{KN}(\alpha)|_\eta)(x) = (A(\alpha) \circ \Psi_{n,A}|_\eta)(x)$$

for all $x \in \mathcal{N}_p(A)^\eta$. By definition of $\Psi_{-,A}$, this condition is written out as:

$$\begin{aligned}
(\Psi_{m,A}|_{\eta'} \circ \mathcal{KN}(\alpha)|_\eta)(x) &= (A(\alpha) \circ \Psi_{n,A}|_\eta)(x) \\
\iff A(\eta')(\mathcal{KN}(\alpha)|_\eta(x)) &= (A(\alpha) \circ A(\eta))(x) && \text{by Eq. (6)} \\
\iff A(\eta')(\mathcal{KN}(\alpha)|_\eta(x)) &= A(\eta\alpha)(x) && \text{by contravariance of } A \\
\iff A(\eta')(\mathcal{KN}(\alpha)|_\eta(x)) &= A(\epsilon\eta')(x) && \text{by the epi-monic factorization of } \eta\alpha \\
\iff A(\eta')(\mathcal{KN}(\alpha)|_\eta(x)) &= A(\eta')(A(\epsilon)(x)) && \text{by contravariance of } A \\
\iff \mathcal{KN}(\alpha)|_\eta(x) &= A(\epsilon)(x) && \text{by injectivity of } A(\eta')
\end{aligned}$$

Recall the definition of structure maps $\mathcal{KN}(\alpha)|_\eta$ (5.1). The maps are defined according to ϵ in this case. Observe that if $\epsilon = \mathbb{1}_{[p]}$, then the condition is satisfied. If $\epsilon = \epsilon_p$, the p th coface map, $A(\epsilon)$ is the p th face map (see Remark 5.1) which corresponds to the differential ∂ of the chain complex, hence again, the p th face map by the isomorphism in Proposition 5.9 (see in particular Eq. (5)). And lastly, both sides are zero if ϵ is any other map. Therefore, the square commutes, hence Ψ_A is a simplicial map.

Given another simplicial map $f : A \rightarrow B$ naturality of Ψ for any two simplicial abelian groups A and B is equivalent to the condition of f commuting with any face or degeneracy morphism by the following diagram:

$$\begin{array}{ccc}
\mathcal{KN}_n(A) & \xrightarrow{\Psi_{n,A} = \bigoplus_\eta A(\eta)} & A_n \\
\mathcal{KN}_n(f) \downarrow & & \downarrow f_n \\
\mathcal{KN}_n(B) & \xrightarrow{\Psi_{n,B} = \bigoplus_\eta B(\eta)} & B_n
\end{array}$$

which is true by definition of a simplicial map.

To see how $N\Psi_A$ is an isomorphism, observe that $\mathcal{N}_n(\mathcal{KN}(A)) = \mathcal{N}_n(A)^{\mathbb{1}_{[n]}}$ so that given any $x \in \mathcal{N}_n(A)^{\mathbb{1}_{[n]}}$, the map Ψ_A acts on it by: $\Psi_{n,A}(x) = A(\mathbb{1}_{[n]})(x) = x$ after which $\mathcal{N}\Psi_A$ is the identity, particularly an isomorphism. Then by Lemma 5.11, it follows that Ψ_A is an isomorphism for all $A \in s\mathbf{Ab}$. $\square$

Hence the proof of category equivalence of Dold-Kan follows.

Proof of Theorem 2.7. The functors $\mathcal{N}$ and $\mathcal{K}$ are given in Propositions 5.4 and 5.8, with natural isomorphisms are shown in Propositions 5.10 and 5.10. By definition of category equivalence (A.1), it follows. $\square$

5.2 Proof of Additional Statements

This subsection studies the consequences of Dold-Kan to, in broad terms, to homotopy and homology.

Homotopy in $s\mathbf{Ab}$ is the same as going to $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ by $\mathcal{N}$ and taking homology. The proof of this statement is adapted from [May67], Proposition 17.4.

Proposition 5.13. *Let A be a simplicial abelian group, then $\pi_*(A) \cong H_*(\mathcal{N}A)$.*

Proof. First, observe that, for fixed $q \geq 0$, the group of q -loops $\tilde{A}_q$ (see (2)) is the same as the set of q -cycles of the normalized complex, $Z_q(\mathcal{N}A)$, (see Definitions 4.2 and 5.4); i.e. $\tilde{A}_q = Z_q(\mathcal{N}A)$. So for any $x \in \tilde{A}_q$, x is a representative of some class in $H_q(\mathcal{N}A)$.

Next, observe that given $x \in B_q(\mathcal{N}A)$, there exists $y \in \mathcal{N}_{q+1}A$ s.t. $\partial_{q+1}y = x$, equivalently $d_{q+1}y = (-1)^{q+1}x$. With $d_iy = 0$ for all $i \leq q$ as $y \in \mathcal{N}_{q+1}A$, y is a homotopy $y : x \sim 0$ (see Definition 3.1). In other words, $x \in B_q(\mathcal{N}A) \iff x \sim 0$.

Then by definitions, we have:

$$\pi_q(A) = \tilde{A}_q / \sim \cong Z_q(\mathcal{N}A) / B_q(\mathcal{N}A) = H_q(\mathcal{N}A)$$

This proves the additional statement in Dold-Kan (5.1) about homotopy $\leftrightarrow$ homology. $\square$

The proof of the following proposition is an expansion of the proof of Lemma 8.3.13 of [Wei94].

Proposition 5.14. *Let $f, g : A \rightarrow B$ be two simplicially homotopic maps. Then $\mathcal{N}f, \mathcal{N}g : \mathcal{N}(A) \rightarrow \mathcal{N}(B)$ are chain homotopic maps.*

Proof. Let $f, g : A \rightarrow B$ be as given with homotopy maps h_j . Since simplicial homotopy is an additive equivalence relation (Thm. 3.5), we can assume $f = 0$. Then, $d_0h_0 = 0$ and $d_{n+1}h_n = g$ according to Def. 3.7. Define $S_n = \sum_{j=0}^n (-1)^j h_j : A_n \rightarrow B_{n+1}$. First, we want to show that $S_n|_{Z_n(A)} \subseteq Z_{n+1}(B)$. Take $x \in Z_n(A)$, which means $d_kx = 0$ for all $k < n$. For any d_i with $0 \leq i \leq n$, observe:

$$\begin{aligned} d_i S_n(x) &= \sum_{j=0}^n (-1)^j d_i h_j(x) \\ &= (-1)^{i-1} d_i h_{i-1}(x) + (-1)^i d_i h_i(x) && \text{via identities in Definition 3.7} \\ &= 0 && \text{via identity } d_i h_i = d_i h_{i-1} \end{aligned}$$

which means, $x \in Z_{n+1}(B)$. Second, d_{n+1} is evaluated on $S_n(x)$:

$$\begin{aligned}
 d_{n+1}S_n(x) &= \sum_{j=0}^{n-1} (-1)^j d_{n+1}h_j(x) + (-1)^n d_{n+1}h_n(x) \\
 &= \left(\sum_{j=0}^{n-1} (-1)^j h_j d_n(x) \right) + (-1)^n g(x) && d_{n+1}h_n = g \text{ and } d_i h_j = h_j d_{i-1} \\
 &= S_{n-1}d_n(x) + (-1)^n g(x) \\
 &\iff d_{n+1}S_n - S_{n-1}d_n = (-1)^n g
 \end{aligned} \tag{7}$$

Finally, we appropriate this to be a chain homotopy. Recall the differential $\partial_n = (-1)^n d_n$ of the normalized chain complex. Observe the chain homotopy condition $\partial S + S\partial$:

$$\partial_{n+1}S_n + S_{n-1}\partial_n = (-1)^{n+1}d_{n+1}S_n + S_{n-1}(-1)^n d_n = (-1)^{n+1}(d_{n+1}S_n - S_{n-1}d_n)$$

Substitute the identity obtained in Eq. (7) to the RHS:

$$\partial_{n+1}(-S_n) + (-S_{n-1})\partial_n = g$$

Thus $-S_n$ is a chain homotopy from $\mathcal{N}0$ to $\mathcal{N}g$. □

Proposition 5.15. *Let $f, g : C \rightarrow C'$ be two chain homotopic maps between non-negatively graded chain complexes C, C' . Then the maps $K(f), K(g) : K(C) \rightarrow K(C')$ are simplicially homotopic.*

Proof. The proof for the converse of the map correspondence is not studied here for saving space. It is as direct as the proof of the forward proposition. See [Wei94], 8.4.4. □

Corollary 5.16. *Under Dold-Kan, simplicially homotopic maps in $s\mathbf{Ab}$ correspond to chain homotopic maps in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$.*

Ultimately, all statements in the Dold-Kan theorem (5.1) have been covered.

6 Application: Eilenberg-MacLane Spaces $K(G, n)$

There is a specific type of space that is of interest in algebraic topology, called Eilenberg-MacLane spaces. As the name suggests, these are due to Eilenberg and MacLane,

from their Annals papers, *On the Groups $H(\Pi, n)$: I, II, and III* ([EL53], [EM54a], [EM54b]). They are of special importance for numerous reasons. Firstly, by the Brown Representability Theorem, they represent (ordinary⁶) cohomology of CW-complexes ([Hat02], 4.E). Second, by Postnikov towers, they are the building blocks of the homotopy type of a space ([Hat02], 4.3). Third, they characterize cohomology operations, first observed by Serre ([Ser53]). The list probably goes on. In this section we mainly study two things: prove their existence via Dold-Kan, and explore arguments related to their homotopy-uniqueness; both in **Top**.

Before giving the definition, we need more results on the connection between simplicial homotopy and topology. The outline given in the introduction will be extended as follows.

$$\begin{array}{ccc}
 \text{Simplicial Homotopy in } s\mathbf{Ab} & \xleftarrow{\text{Dold-Kan}} & \text{Homological Algebra in } \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) \\
 \cap & & \\
 \text{Simplicial Homotopy in } s\mathbf{Set} & & \\
 \downarrow \scriptstyle{|-|} \quad \uparrow \scriptstyle{\mathcal{S}(-)} & & \\
 \text{Topology in } \mathbf{Top} & &
 \end{array}$$

6.1 Further on Realization

Our first discussion of realization, in Section 2.3, mentioning points of $|X|$ in *indexed geometric simplices* $x \times |\Delta^n|$ begs the question: After identifications, what kind of a structure do we get? Does realization produce something *nice*? Is there an inverse functor to realization? The results that follow answer all of these questions. For a rather expository motivating example for these results, see Appendix A.3. The results in this subsection are given mostly without proof for space constraints and as their proofs are peripheral to the focus of this project.

Theorem 6.1 ([May67], Theorem 14.1). Given a simplicial set X , $|X|$ is a CW-complex having one n -cell for each non-degenerate n -simplex of X .

⁶Full satisfaction of the Eilenberg-Steenrod axioms.

Theorem 6.2 ([May67], Theorem 14.3 and Remark 14.4). Let K and L be two simplicial sets. Define $\theta : |K \times L| \rightarrow |K| \times |L|$ via realization of the projection maps. Then, θ is a bijection and in particular, if $|L|$ is locally finite, θ is a homeomorphism.

Now we can carry simplicial homotopies to topological homotopies.

Proposition 6.3. *The simplicial homotopies in $s\mathbf{Set}$ can be carried over to topological homotopies in $\mathbf{Top}$.*

Proof. Recall the product description of a simplicial homotopy in Definition 3.6. Take one, $H : X \times \Delta^1 \rightarrow Y$ where $X, Y \in s\mathbf{Set}$. Since $|\Delta^1| \cong I$ is trivially locally finite, $|X \times \Delta^1|$ is homeomorphic to $|X| \times |\Delta^1|$ by Theorem 6.2 so that $|H| \circ \theta^{-1} : |X| \times |\Delta^1| \rightarrow |Y|$ is a homotopy in $\mathbf{Top}$. $\square$

Recall the singular set $\mathcal{S}(X)$ associated to a topological space X (see Example 2.4). The question of “An inverse to realization?” is still unanswered so far in this project. Sadly, such a thing does not exist. However, we have the following.

Proposition 6.4 ([May67], Theorem 16.1). *Realization and the singular set form functors:*
 $| - | : s\mathbf{Set} \rightleftarrows \mathbf{Top} : \mathcal{S}(-)$

Theorem 6.5 ([May67], Theorem 16.6). Let $(K, *)$ be a Kan pair (c.f. Def. 3.2) Then:

$$\pi_n(K, *) \cong \pi_n(|K|, |*|)$$

In particular, this subsection results in the fact that we can move homotopies and homotopy groups from Kan complexes in $s\mathbf{Set}$ to $\mathbf{Top}$.

6.2 Definition of $K(G, n)$

In this subsection, we define $K(G, n)$ and briefly refer to $K(G, n)$ in $\mathbf{Top}$ to show the difficulty and *inelegance* of its construction.

Definition 6.1. A topological space X having only one nontrivial homotopy group $\pi_n(X) \cong G$ at n is called an **Eilenberg-MacLane space of type $K(G, n)$** .

By the results from Subsection 6.1, this definition is the same for Kan complexes of $s\mathbf{Set}$.

Definition 6.2. A Kan complex K having only one nontrivial homotopy group $\pi_n(X) \cong G$ at $n \geq 1$ is called an Eilenberg-MacLane space of type $K(G, n)$.

The reader is referred to [Hat02], Section 4.2, p. 365, to see how tedious it is to construct a general $K(G, n)$ which is done by attaching disks along some spheres onto wedges of n -spheres to first create the desired homotopy groups, then kill off the pathologically produced ones. That is why the following existence proof can be thought of as elegant.

6.3 Proof of Existence of $K(G, n)$ in **Top**

Proposition 6.6. *Eilenberg-MacLane spaces of type $K(G, n)$ exist in **Top** for any $n \geq 2$ and for any abelian group G .*

Proof. Let $C = G[-n]$ be the chain complex that is G concentrated in degree n . I.e. $C_k = 0$ for all $k \neq n$ and $C_n = G$:

$$\dots \xrightarrow{0} 0 \xrightarrow{0} C_{n+1} = 0 \xrightarrow{0} C_n = G \xrightarrow{0} C_{n-1} = 0 \xrightarrow{0} \dots \xrightarrow{0} C_0 = 0$$

The only nonzero homology group of this complex is $H_n(C) = G$. Here is the magic. Consider the simplicial abelian group $A = \mathcal{K}(C)$ where $\mathcal{K}$ is the inverse Dold-Kan functor. Since homotopy $\leftrightarrow$ homology by Proposition 5.13, A is of type $K(G, n)$ in **sAb**. Consider the space $X = |A|$. By Theorems 6.1 and 6.5, X is of type $K(G, n)$ in **Top**. $\square$

6.4 On Uniqueness of $K(G, n)$ up to Homotopy Equivalence

In this subsection, we investigate uniqueness of $K(G, n)$ in **Top** up to homotopy equivalence. We first attempt to use Dold-Kan as instructed by Exercise 8.4.4.(2) of [Wei94], then explain the limitation of Dold-Kan for the purpose of uniqueness.

Proposition 6.7. *Let A be any simplicial abelian group of type $K(G, n)$ and C be the chain complex that is G concentrated in degree n as described in the proof of Proposition 6.6. Then $|A|$ is homotopy equivalent to $|K(C)|$.*

Before proving this, a result is needed.

Definition 6.3. A map $f : X \rightarrow Y$ of spaces is called a **weak homotopy equivalence** if $f_\# : \pi_i(X) \rightarrow \pi_i(Y)$ is an isomorphism for all $i \geq 1$. If such a map exists, X and Y are called weak homotopy equivalent, denoted by $X \simeq_{\text{weak}} Y$.

The chain complex parallel of weak homotopy equivalence is defined as follows.

Definition 6.4. A chain map $P \rightarrow W$ is called a **quasi-isomorphism** if the maps $H_n(P) \rightarrow H_n(W)$ (maps from the homology functor; see [Bre97], Corollary 4.2) are isomorphisms for all n (over the range on which the chain complexes are defined). Quasi-isomorphic complexes are denoted by $P \simeq_{\text{quasi}} Q$.

Definition of weak homotopy equivalence is the same in $s\mathbf{Set}$ on Kan complexes as objects and maps are in $s\mathbf{Set}$ instead.

Theorem 6.8 ([Bre97], Corollary 11.14). Let $f : X \rightarrow Y$ be a map between connected CW complexes. Then, f is a homotopy equivalence if and only if f is a weak homotopy equivalence.

Proof of Proposition 6.7. Let A and C be as given. Consider the pair A and $\mathcal{K}(C)$ as simplicial abelian groups, they are both of type $K(G, n)$. The first idea that comes to mind would be to establish somehow a simplicial homotopy and carry it right away to **Top** using Proposition 6.3, however this would be too much effort compared to using Dold-Kan and chain complexes.

Apply the normalized chain complex functor $\mathcal{N}$ to A to obtain $(\mathcal{N}(A), \partial)$. From Proposition 5.13, $H_n(\mathcal{N}(A)) \cong G$ and $H_k(\mathcal{N}(A)) = 0$ for all $k \neq n$, as A is of type $K(G, n)$. Construct the following subcomplex V of $\mathcal{N}(A)$ by:

$$V_i := \begin{cases} \mathcal{N}_i(A), & \text{if } i < n; \\ Z_n(\mathcal{N}(A)), & \text{if } i = n; \\ 0, & \text{if } i > n \end{cases}$$

where ∂^V is given by the restriction of $\partial^{\mathcal{N}(A)}$. By construction, we have $H_n(\mathcal{N}(V)) \cong G$ and $H_k(\mathcal{N}(V)) = 0$. Then, the chain complex inclusion map $i : V \hookrightarrow \mathcal{N}(V)$ is a quasi-isomorphism. Next, construct a map $p : \mathcal{N}(V) \rightarrow C$ as $p_i \equiv 0$ for all $i \neq n$ and $p_n : Z_n(\mathcal{N}(V)) \rightarrow C$ as the canonical projection map onto the homology group, i.e. $p_n(x) = x + B_n(\mathcal{N}(V))$. Checking whether p is a valid map is simple, we just need the following commutativity: $p_n(\partial_{n+1}x) = \partial_{n+1}x + B_n(\mathcal{N}(V))$. Since $\partial_{n+1}x \in \text{im}(\partial_{n+1}) = B_n(\mathcal{N}(V))$, it maps to the zero element in the quotient group G . This matches the zero map from $C_{n+1} \rightarrow 0$, so p is a chain map. Since this single map induces an isomorphism

on homology groups, it is trivially a quasi-isomorphism. Then we have the following zig-zag of quasi-isomorphism:

$$\mathcal{N}(A) \xleftarrow{i} V \xrightarrow{p} C$$

Since $\mathcal{K}$ is an equivalence of categories, we can apply it to the zig-zag above to obtain a zig-zag of weak homotopy equivalences of Kan complexes A and $\mathcal{K}(C)$:

$$A \xleftarrow{\simeq_{\text{quasi}}} \mathcal{K}(V) \xrightarrow{\simeq_{\text{quasi}}} \mathcal{K}(C)$$

We can apply the realization functor to this zig-zag. By Theorem 6.1, the objects are nice CW complexes. By Theorem 6.5 and functoriality we obtain the zig-zag of topological weak homotopy equivalences, which, by Theorem 6.8, upgrades to homotopy equivalence as follows:

$$\begin{array}{ccc} |A| & \xleftarrow{\simeq_{\text{weak}}} |\mathcal{K}(V)| & \xrightarrow{\simeq_{\text{weak}}} |\mathcal{K}(C)| \\ & \searrow \quad \quad \nearrow & \\ & \simeq & \end{array}$$

□

The kind of uniqueness we had shown is really restricted. This is due to the fact that very few topological spaces correspond to a simplicial abelian group, under the singular set functor $\mathcal{S}(-)$ (see Proposition 6.4). So what about the question of uniqueness in general for any arbitrary space of type $K(G, n)$. Is it homotopy equivalent to the construction given by Dold-Kan? The answer is yes, see [May67], Theorem 23.9. However, one either needs to do topological obstruction theory or simplicial obstruction theory, as May does it in the latter way, to construct the map needed to invoke Theorem 6.8.

A Appendices

A.1 Category Theoretic Notions

The material in this appendix is from [Rie16].

Theorem A.1 (Yoneda lemma). For any functor $\mathcal{F} : C \rightarrow \mathbf{Set}$ on a locally small category C , and for any object c in C , there exists a bijection:

$$\mathrm{Hom}(C(c, -), \mathcal{F}) \cong \mathcal{F}_c$$

that associates a natural transformation $\alpha : C(c, -) \Rightarrow \mathcal{F}$ to the element $\alpha_c(1_c) \in \mathcal{F}_c$ and this correspondence is natural in both c and $\mathcal{F}$.

Some authors use Nat instead of Hom to denote the fact that we have natural transformations but it is exactly the same as the hom-set in the functor category $\mathbf{Set}^C$.

Its version for contravariant functors is as follows

Theorem A.2 (Yoneda lemma for contravariant $\mathcal{F}$). For any functor $\mathcal{F} : C^{op} \rightarrow \mathbf{Set}$ on a locally small category C , and for any object c in C , there exists a bijection:

$$\mathrm{Hom}(C(-, c), \mathcal{F}) \cong \mathcal{F}_c$$

that associates a natural transformation $\alpha : C(-, c) \Rightarrow \mathcal{F}$ to the element $\alpha_c(1_c) \in \mathcal{F}_c$ and this correspondence is natural in both c and $\mathcal{F}$.

As a corollary, we have the covariant and contravariant Yoneda embeddings.

Corollary A.3 (Yoneda embeddings). *The following functors define full and faithful embeddings:*

$$\begin{array}{ccc} C & \xhookrightarrow{y} & \mathbf{Set}^{C^{op}} \\ c & \mapsto & C(-, c) \\ f \downarrow & \longmapsto & \downarrow f_* \\ d & \mapsto & C(-, d) \end{array} \qquad \begin{array}{ccc} C^{op} & \xhookrightarrow{y} & \mathbf{Set}^C \\ c & \mapsto & C(c, -) \\ f \downarrow & \longmapsto & \uparrow f_* \\ d & \mapsto & C(d, -) \end{array}$$

Definition A.1. An **category equivalence** between categories C and D consists of functors $F : C \rightleftarrows D : G$ together with natural isomorphisms

$$\eta : 1_C \cong GF \quad \text{and} \quad \epsilon : FG \cong 1_D$$

equivalence denoted with $C \simeq D$.

Definition A.2. Let C, D be categories and $F, G : C \rightrightarrows D$ be functors. Then a **natural transformation** $\alpha : F \Rightarrow G$ consists of a collection of morphisms $\{\alpha_c : Fc \rightarrow Gc\}$ where is morphism α_c is for each object $c \in C$ so that the morphisms $\{\alpha_c\}$ are natural in C . In other words, given any morphism $f : x \rightarrow y$ in C , the following square commutes:

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha_x} & Gx \\ Ff \downarrow & & \downarrow Gf \\ Fy & \xrightarrow{\alpha_y} & Gy \end{array}$$

A natural transformation $\alpha : F \Rightarrow G$ is called a **natural isomorphism** of functors F and G if every morphism α_c is an isomorphism in C .

Proposition A.4 (Density Theorem for Simplicial Sets; [Rie11], p. 5). *Every simplicial set X is canonically isomorphic to a colimit of standard simplices. Specifically, it is the colimit over its category of elements $\Delta \downarrow X$:*

$$X \cong \varinjlim_{(\Delta[n] \xrightarrow{x} X) \in \Delta \downarrow X} \Delta[n]$$

A.2 Proofs of Well-Definition of $\mathcal{C}$ and $\mathcal{D}$

Proof of Lemma 5.2. Given a simplicial abelian group A , let $\mathcal{C}(A)$ be as defined. Showing well-definition of $\mathcal{C}$ amounts to showing that $\mathcal{C}(A)$ is a chain complex and $\mathcal{C}(f)$ is a chain map for any simplicial map f . For the former, one needs to show that $\partial^{\mathcal{C}}$ is a well-defined differential on $\mathcal{C}(A)$; equivalently, one needs to show closure, homomorphism, and being of order two.

Since addition and inverses are well-defined in any A_n , the alternating sum $\sum (-1)^i d_i$ is a well-defined $(n-1)$ -simplex in A_{n-1} . Homomorphism follows by $\mathbb{Z}$ -linearity. To see

why ∂ is of order two, observe:

$$\begin{aligned}
\partial_n \circ \partial_{n+1} &= \partial_n \left(\sum_{i=0}^{n+1} (-1)^i d_i \right) \\
&= \sum_{j=0}^n (-1)^j d_j \left(\sum_{i=0}^{n+1} (-1)^i d_i \right) \\
&= \sum_{j=0}^1 \sum_{i=0}^{n+1} (-1)^{i+j} d_j d_i && \text{by } \mathbb{Z}\text{-linearity,} \\
&= \sum_{0 \leq j < i \leq n+1} (-1)^{i+j} d_j d_i + \sum_{0 \leq i \leq j < n} (-1)^{i+j} d_j d_i && \text{decomposing,} \\
&= \sum_{0 \leq j < i \leq n+1} (-1)^{i+j} d_{i-1} d_j + \sum_{0 \leq i \leq j < n} (-1)^{i+j} d_j d_i && \text{by simplicial identity for } d_j d_i, \\
&= - \sum_{0 \leq j \leq i < n} (-1)^{i+j} d_i d_j + \sum_{0 \leq j \leq i < n} (-1)^{i+j} d_j d_i && \text{reindexing } j \text{ for } j+1, \\
&= 0
\end{aligned}$$

Therefore $\partial \circ \partial \equiv 0$. Hence $(\mathcal{C}(A), \partial)$ is a chain complex. Functoriality is immediate as everything takes place back in $S\mathbf{Ab}$; a simplicial map f is mapped to a chain map $\mathcal{C}(f)$ i.e. it commutes with the differential, because via naturality and $\mathbb{Z}$ -linearity, simplicial maps commute with alternating sums of faces - which is the differential. $\square$

Proof of Lemma 5.3. To see that $\mathcal{D}$ has well-defined assignments, observe that images of degeneracies are subgroups of $\mathcal{C}_n(A)$ as s_i are group homomorphisms. And with the fact that sum of subgroups is a subgroup, $\mathcal{D}_{n-1}(A)$ is a subgroup of $C_n = A_n$. Particularly, it is an abelian group by itself which makes up a chain complex with ∂ , shown to be a differential in a minute. For morphisms, a simplicial morphism is mapped to a chain map for the same reason as in the proof of Lemma 5.2.

For functoriality of $\mathcal{D}$, composition law follows from the fact that simplicial maps commute with degeneracies. I.e. given a simplicial map $f : A \rightarrow B$ of simplicial abelian groups, we have the following assignment at each n :

$$\begin{array}{ccc}
\mathcal{D}(A)_n & \hookrightarrow & A_n \\
\mathcal{D}(f) \downarrow & & \downarrow f \\
\mathcal{D}(B)_n & \hookrightarrow & B_n
\end{array}$$

And given another map $g : B \rightarrow C$, since g and f are composable as $g \circ f$, and as maps

g and f commute with degeneracies by their definition, one has $\mathcal{D}(g \circ f) = \mathcal{D}(g) \circ \mathcal{D}(f)$. $\mathcal{D}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{D}(A)}$ is by definition.

For well-definition of $\partial^{\mathcal{D}}$, homomorphism again follows from $\mathbb{Z}$ -linearity. For verifying the codomain, recall that $\mathcal{D}_n(A)$ is a subgroup of C_n , particularly is closed under additions. Thus, to show the well-definition of $\partial^{\mathcal{D}}$, it suffices to show where it lands for a single degeneracy $y = s_j(x) \in \mathcal{D}_n(A)$ for $x \in C_{n-1}(A)$ and $0 \leq j \leq n-1$.

$$\begin{aligned}
 \partial(y) &= \sum_{i=0}^n (-1)^i d_i s_j(x) \\
 &= \sum_{i=0}^{j-1} (-1)^i d_i s_j(x) + (-1)^j d_j s_j(x) + (-1)^{j+1} d_{j+1} s_j(x) + \sum_{i=j+2}^n (-1)^i d_i s_j(x) \\
 &= \sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i(x) + (-1)^j x - (-1)^j x + \sum_{i=j+2}^n (-1)^i s_j d_{i-1}(x) \\
 &= s_{j-1} \left(\sum_{i=0}^{j-1} (-1)^i d_i(x) \right) + 0 + s_j \left(\sum_{i=j+2}^n (-1)^i d_{i-1}(x) \right) \in \mathcal{D}_{n-1}(A)
 \end{aligned}$$

Third equality follows from simplicial identities (see Definition 2.1). Lastly, $\partial^{\mathcal{D}}$ is of order two, thus a valid differential map, directly from the proof of Lemma 5.2. $\square$

A.3 A Motivating Example for Realization

This appendix is to keep an example that is immediately applicable to the theory develop after Dold-Kan, to motivate the serious results on realization. This example is adapted from [Fri23].

Remark A.1. Let X be an ordered simplicial complex and $\overline{X}$ be its associated simplicial set. Then $X = |\overline{X}|$.

Example A.1. We can use the previous remark to obtain the $(n-1)$ -sphere as the realization of the simplicial set $\partial\Delta^n$ associated to the topological boundary of the ordered standard n -simplex, $|\Delta^n|$, as follows:

$$\begin{array}{ccccccc}
 & \text{Take top.} & & \text{Adj. all degen.} & & & \\
 & \text{boundary} & & \text{simplices} & & \text{Realize} & \\
 |\Delta^n| & \xrightarrow{\quad} & \partial|\Delta^n| & \xrightarrow{\quad} & \overline{\partial|\Delta^n|} =: \partial\Delta^n & \xrightarrow{\quad} & |\partial\Delta^n| \\
 \vdots & & & & & & \vdots \\
 n\text{-simplex} & & & & & & (n-1)\text{-sphere}
 \end{array}$$

Merely considering the fact that $\partial\Delta^n$ takes its simplices from Δ^n except the full simplex Δ^n , it is apparent that it is going to have a lot of simplices. So, a more efficient way of obtaining the $(n-1)$ -sphere would be to *imitate* the CW-complex construction for $n \geq 2$. Consider X as the following simplicial set:

$$X = \begin{cases} [0] \in X_0 \\ [0, \underbrace{\dots, 0}_{i \text{ times}}] \in X_i \text{ for } 0 < i < n-1 \\ [0, 1, \dots, n-1] \in X_{n-1} \end{cases}$$

This construction determines the face and degeneracy maps. It can be seen that the realization $|X|$ is equivalent to the standard CW complex construction of S^{n-1} obtained by collapsing the boundary of an $(n-1)$ -cell to a point (see [Hat02], Example 0.3).

On how nice the realization procedure is and what kind of a structure it produces, from Example A.1, we observe the following:

- (i) Simplicial realization does not necessarily result in a simplicial complex structure, seen by the second construction of $|\partial\Delta^n|$,
- (ii) Simplicial realization is not unique in the sense that for two different-looking simplicial sets X and Y , we can have $|X| \cong |Y|$,
- (iii) Simplicial realization can be understood only by looking at what happens to non-degenerate simplices, as degenerate ones get collapsed to simplices of which they are degeneracies.

These observations further clarify realization beyond its definition and examples in Subsection 2.3.

References

- [Bre97] GLEN E. BREDON. *Topology and geometry*. Vol. 139. Graduate Texts in Mathematics. Corrected third printing of the 1993 original. Springer-Verlag, New York, 1997, pp. xiv+557. ISBN: 0-387-97926-3 (cited p. 42)
- [EL53] SAMUEL EILENBERG and SAUNDERS MAC LANE. On the Groups $H(\Pi, n)$, I. In: *Annals of Mathematics*, **58**:1 (1953), pp. 55–106. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1969820> (visited on 05/05/2026) (cited p. 39)
- [EM54a] SAMUEL EILENBERG and SAUNDERS MACLANE. On the Groups $H(\Pi, n)$, II: Methods of Computation. In: *Annals of Mathematics*, **60**:1 (1954), pp. 49–139. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1969702> (visited on 05/05/2026) (cited p. 39)
- [EM54b] SAMUEL EILENBERG and SAUNDERS MACLANE. On the Groups $H(\Pi, n)$, III: Operations and Obstructions. In: *Annals of Mathematics*, **60**:3 (1954), pp. 513–557. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1969849> (visited on 05/05/2026) (cited p. 39)
- [Fri23] GREG FRIEDMAN. *An elementary illustrated introduction to simplicial sets*. 2023. arXiv: 0809.4221 [math.AT]. URL: <https://arxiv.org/abs/0809.4221> (cited pp. 3, 11, 16–18, 31, 47)
- [GJ09] PAUL G. GOERSS and JOHN F. JARDINE. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Reprint of the 1999 edition [MR1711612]. Birkhäuser Verlag, Basel, 2009, pp. xvi+510. ISBN: 978-3-0346-0188-7. DOI: [10.1007/978-3-0346-0189-4](https://doi.org/10.1007/978-3-0346-0189-4). URL: <https://doi.org/10.1007/978-3-0346-0189-4> (cited pp. 10, 21, 27, 31)
- [Hat02] ALLEN HATCHER. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0 (cited pp. 15, 39, 41, 48)
- [Hug22] CALUM HUGHES. *The Dold-Kan Correspondence*. Master’s thesis. University of Sheffield, 2022 (cited pp. 21, 33)

- [Kan55] DANIEL M. KAN. Abstract Homotopy. I. In: *Proceedings of the National Academy of Sciences of the United States of America*, **41**:12 (1955), pp. 1092–1096. ISSN: 00278424, 10916490. URL: <http://www.jstor.org/stable/89108> (visited on 03/29/2026) (cited p. 12)
- [May67] J. PETER MAY. *Simplicial objects in algebraic topology*. Vol. No. 11. Van Nostrand Mathematical Studies. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967, pp. vi+161 (cited pp. 12, 15, 16, 18, 31, 37, 39, 40, 43)
- [MKP25] JAMES R. MUNKRES, STEVEN G. KRANTZ, and HAROLD R. PARKS. *Elements of algebraic topology*. Second. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2025, pp. xiii+569. ISBN: 978-1-032-76554-9; 978-1-041-02924-3; 978-1-003-62147-8 (cited p. 3)
- [Moo58] JOHN C. MOORE. Semi-simplicial complexes and Postnikov systems. In: *1958 Symposium internacional de topología algebraica International symposium on algebraic topology*, (1958). Unpublished, pp. 232–247 (cited p. 14)
- [Mun75] JAMES R. MUNKRES. *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975, pp. xvi+413 (cited p. 13)
- [Rie11] EMILY RIEHL. “A Leisurely Introduction to Simplicial Sets”. In: 2011. URL: <https://api.semanticscholar.org/CorpusID:5055150> (cited p. 45)
- [Rie16] EMILY RIEHL. *Category theory in context*. Aurora Dover Modern Math Originals. Dover Publications, Inc., Mineola, NY, 2016, pp. xvii+240. ISBN: 978-0-486-80903-8; 0-486-80903-X (cited p. 44)
- [Ser53] JEAN-PIERRE SERRE. Cohomologie modulo 2 des complexes d’Eilenberg-MacLane. In: *Comment. Math. Helv.*, **27**: (1953), pp. 198–232. ISSN: 0010-2571,1420-8946. DOI: [10.1007/BF02564562](https://doi.org/10.1007/BF02564562). URL: <https://doi.org/10.1007/BF02564562> (cited p. 39)
- [Wei94] CHARLES A. WEIBEL. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994 (cited pp. 7, 10, 14, 17, 21, 26, 28, 31, 34, 37, 38, 41)